6. How many ordered triplets (a, b, c) of non – zero real numbers have the property that each number is the product of the other two?
There are **four** ordered triplets: (1, 1, 1), (-1, -1, 1), (-1, 1, -1), (1, -1, -1)

7. Rajat decided to tell the truth on Mondays, Thursdays and Saturdays, but lie on every other day. One day he says, "I will tell the truth tomorrow." What day of the week he made this statement?
His statement can't be true as in that case he'd have been speaking a truth for two consecutive days which is contrary to given information. Hence he is telling a lie and next day also he's going to lie. Two consecutive days on which he can tell a lie are Tuesday and Wednesday only. Hence he made the above statement on **Tuesday**.

8. For what smallest positive integral n, factorial of n is divisible by 414?
$414 = 2 \times 3^2 \times 23$. So n! must have atleast one 2, two 3's and one 23. Smallest possible n is **23**.

9. 2010 inhabitants of TG Land are divided into two groups: the Truth tellers – who always tell the truth and the Liars – who always tell a lie. Each person is exactly one of the following – a cricketer, a guitarist or a swimmer. Each inhabitant was asked the three questions: 1) Are you a cricketer? 2) Are you a guitarist? 3) Are you a swimmer? 1221 persons answered "yes" to the first question. 729 persons answered "yes" to second question and 660 persons answered "yes" to third question. How many "Liars" are present on the TG Land?
If each inhabitant must have spoken truthfully, then there should be 2010 'yes' in the response to the three questions. But due to each liar number of 'yes' will increase by one (because when the person was telling truth, he must have said one 'yes' and two 'no' but if he will lie then he must say two 'yes' and one 'no').
Number of 'yes' received in response to three questions are: 1221 + 729 + 660 = 2610 which is 600 more than 2010. Hence **600** liars are present on the TG Land.

10. Isosceles triangle ABC has the property that, if D is a point on AC such that BD bisects angle ABC, then triangle ABC and BCD are similar. If BC has length of one unit, then what is the length of AB?
Given that triangle ABC is isosceles as shown in the diagram and also BD is bisector of $\angle ABC = 2\theta$.
So $\angle ABD = \angle CBD = \theta$ and $\angle ACD = 2\theta$ as shown below.
Also given that triangle ABC and BCD are similar so all the three angles of two triangles must be same. In triangle ABC two angles are equal to 2θ so must be in triangle BCD. Hence $\angle BDC = 2\theta$ and also $\angle BAC = \theta$ as shown in diagram above.
Hence triangle BCD is isosceles such that BD = BC = 1. And also triangle ADB is isosceles such that AD = BD = 1. Let CD = x, so that AB = x + 1.
Now equating the ratios of sides in similar triangles ABC and BCD, we have

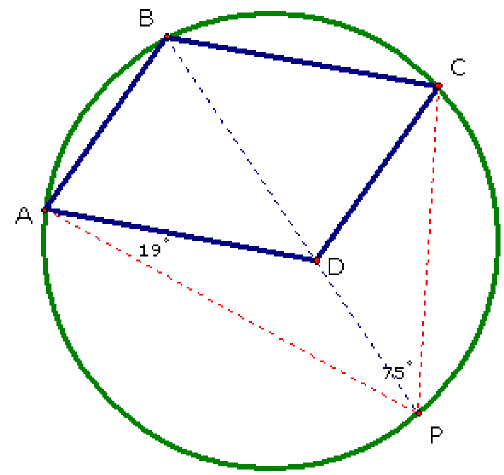
$$\frac{AB}{BC} = \frac{BC}{CD}$$

$$\frac{x+1}{1} = \frac{1}{x}$$

$\Rightarrow x^2 + x - 1 = 0$.

Because x is positive, $x = \frac{\sqrt{5}-1}{2}$ and $\boxed{AB = x + 1 = \frac{\sqrt{5}+1}{2}}$.

11. Vertices A, B and C of a parallelogram ABCD lie on a circle and D lies inside the circle such that line BD intersects the circle at point P. Given that ∠APC = 75° and ∠PAD = 19°, what is the measure of ∠PCD?



∠ABC = 180° - ∠APC = 105° (because ABCP is a cyclic quadrilateral)
Also ∠ADC = ∠ABC = 105° (opposite angles of a parallelogram are equal)
∠ADC = ∠ADB + ∠CDB = (∠PAD + ∠APD) + (∠PCD + ∠CPD)
105° = 19° + 75° + ∠PCD
=> **∠PCD = 11°**

12. Read the following 5 statements carefully:
(i) Statement (ii) is true.
(ii) At most one of the given five statements is true.
(iii) All of the given statements are true.
(iv) .
(v) .

The last two statements are printed in invisible ink. Which of the statements are true?
Both **statements (iv) and (v) are true**.

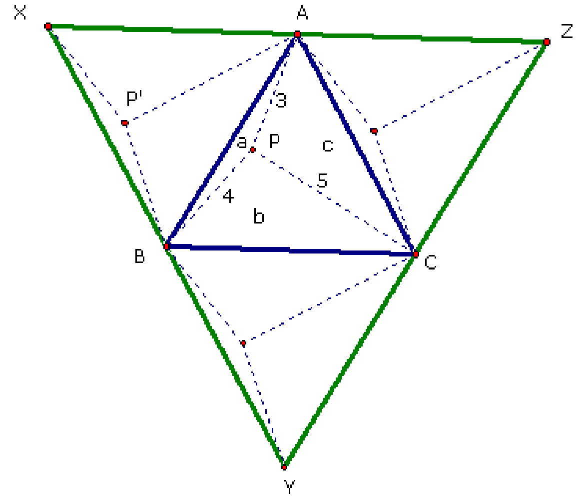
13. While adding all the page numbers of a book, I found the sum to be 1000. But then I realized that two page numbers (not necessarily consecutive) have not been counted. How many different pairs of two page numbers can be there?
Total number of pages can be either 45 as 1 + 2 + 3 +... + 45 = 1035 Or 46 as 1 + 2 + 3 +... + 46 = 1081.
So sum of missing page numbers is 35 or 81.
Now let a and b be the two page numbers. So in first case,
a + b = 35 and a, b ∈ {1, 2... 45} i.e. total 17 cases - (1, 34), (2, 33)... (17, 18)
In second case, a + b = 81 and a, b ∈ {1, 2... 46} i.e. total 6 cases – (35, 46), (36, 45)... (40, 41)
Hence in all, **23** different pairs of two page numbers can be there.

14. How many positive integers are equal to 12 times the sum of their digits?
Let the number have two digits so that N = 10a + b = 12(a + b) => 2a + 11b = 0 which is not possible.
If N is a three digit number, then N = 100a + 10b + c = 12(a + b + c) => 88a = 2b + 11c.
Only possibility is: a = 1, b = 0 and c = 8 i.e. N = **108**.
If N is a four digit number, then N = 1000a + 100b + 10c + d = 12(a + b + c + d)
=> 988a + 88b = 2c + 11d. So RHS can never be greater than 117 and hence no four digit number can satisfy. Similarly any number with more than four digit will not satisfy the given property.
Hence only **One number** is equal to 12 times the sum of its digits.

15. An 8 cm by 12 cm rectangle is folded along its long side so that two diagonally opposite corners coincide. What is the length of crease formed?
This can be solved by making some right angle triangles and using Pythagoras theorem. A more beautiful solution is to observe that crease must form the perpendicular bisector of the diagonal. The length of the diagonal is $\sqrt{8^2+12^2}=4\sqrt{13}$. If x is the length of crease then triangle formed by half the crease and half the diagonal is similar to the triangle formed by the diagonal and two sides of the rectangle.

So, $\frac{x/2}{2\sqrt{13}}=\frac{8}{12}$ => $\boxed{x = \frac{8\sqrt{13}}{3}}$

16. A point P inside an equilateral triangle, ABC is located at a distance of 3, 4 and 5 units respectively from A, B and C. What is the area of the triangle ABC?



Let area of triangle ABC is divided into three triangles of area a, b and c as shown. Now rotate the triangle ABC from vertex B to XBA so that P is shifting to P'(say). Now PBP' is an equilateral triangle of side 4 units and APP' is a right angle triangle with side lengths (3, 4, 5).

So a + b = ar(PBP') + ar(APP') = $\frac{\sqrt{3}}{4}4^2 + 6$.

Similarly b + c = $\frac{\sqrt{3}}{4}5^2 + 6$, and c + a = $\frac{\sqrt{3}}{4}3^2 + 6$.

So area of triangle ABC = $\frac{(a+b)+(b+c)(c+a)}{2} = \frac{\sqrt{3}}{4}5^2 + 9$.

Hence $\boxed{\text{required area } = \frac{36+25\sqrt{3}}{4} \approx 19.8 \text{square units}}$

17. Ten boxes each contain 9 balls. The balls in one box each weigh 0.9 kg; the rest all weigh 1 kg. In how many least number of weighing you can determine the box with the light balls?

Just number the boxes as #0, #1, #2… #9 and now take 0, 1, 2… 9 balls respectively from each box in that order. Now weigh all the 45 balls together in one weighing. Total weight will be (45 – a)kg. 'a' can easily be calculated from the weight obtained. Now box with lighter balls is '10a'. Hence only **one weighing** is sufficient.

18. A given circle has n chords. Each chord crosses every other chord but no three chords meet at the same point. How many regions are in the circle?

Number of region made by n intersecting chord is given by $^{n}C_2 + n + 1$.

19. Find all prime numbers *p* for which *5p + 1* is a perfect square.

Let *5p + 1 = (x + 1)*^{2} *=> 5p = x(x + 2).*

As LHS is product of two prime numbers so must be RHS. Also from RHS it is clear that difference between two prime numbers is 2 only. There are only **two** prime numbers i.e. 3 and 7 which are at a distance of two units from 5.

20. A programmer carelessly increased the tens digit by 1 for each multi-digit Fermat number in a lengthy list produced by a computer program.

Fermat numbers are integers of the form: $N = 2^{2^n} + 1$ **for integer n > 1. How many numbers on this new list are prime?**

By adding 1 to ten's digit we are just adding 10 to the number. So the given number becomes

$$N + 10 = 2^{2^n} + 11 \equiv (1 + 2) \bmod 3 \equiv 0 \bmod 3.$$

Hence all the new numbers are divisible by 3 and **none** is prime.

21. What is the greatest common divisor of the 2010 digit and 2005 digit numbers below?

$$\underbrace{33333...333}_{2010\ 3's} \quad \underbrace{7777...77}_{2005\ 7's}$$

Let A = 333..33 and B = 777..77, then A/3 and B/7 both contains the digit 1 only (2010 times and 2005 times respectively). As HCF(2010, 2005) = 5 that means HCF(A/3, B/7) = 11111.

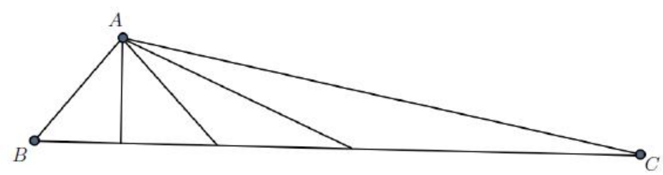
Now A is also divisible by 7 as 2010 = 6 $\times$ 335 (Remember a number formed by repeating same digit 6 times is divisible by 7). So HCF(A, B) = **77777**.

22. Two players play a game on the board below as follows. Each person takes turns moving the letter A either downward at least one rectangle or to the left at least one rectangle (so each turn consists of moving either downward or to the left but not both). The first person to place the letter A on the rectangle marked with the letter B wins. How should the first player begin this game if we want to assure that he wins? Answer with the number given on the rectangle that he should move the letter A to.

1	2	3	4	5	6	7	8	9	**A**
									10
									11
B									12

The first person should begin by moving *A* to the rectangle numbered 4. In fact, the first person can win by, on each turn, putting *A* on a rectangle along the diagonal from the rectangle labelled *B* to the rectangle numbered 4. The second player will have to move off this diagonal and then the first player can continue to move on the diagonal. Since the rectangle labelled with *B* is on the diagonal, the first person will eventually win (with this strategy). Observe that if first player does not put *A* on the rectangle numbered 4, then the second player can force a win by using the above strategy (putting *A* along the diagonal).

23. In ΔABC (not drawn to scale), the altitude from *A*, the angle bisector of $\angle BAC$, and the median from *A* to the midpoint of *BC* divide $\angle BAC$ into four equal angles. What is the measure in degrees of angle $\angle BAC$?



Just apply Sine rule to find that $\angle$**BAC = 90°**.

24. Let a_1, a_2,..a_{2011} represents the arbitrary arrangement of the numbers 1, 2,..2011. Then what is the remainder when $(a_1 - 1)(a_2 - 2)..(a_{2011} - 2011)$ is divided by 2?
We have 1006 odd numbers and 1005 even numbers. So atleast one of the factors, which are multiplied, has to be the difference of two odd numbers and hence even. So the required remainder is **zero**.

25. One side of a triangle has length 75. Of the other two sides, the length of one is double the length of the other. What is the maximum possible area for this triangle?
Let other two sides be x and 2x, then using Heron's formula to find the area of triangle we have,

$$\Delta = \sqrt{\left(\frac{3x+75}{2}\right)\left(\frac{x+75}{2}\right)\left(\frac{-x+75}{2}\right)\left(\frac{3x-75}{2}\right)} = \frac{3}{4}\sqrt{\left(x^2-25^2\right)\left(75^2-x^2\right)}$$

Now area is maximum when $(x^2 - 25^2) = (75^2 - x^2) = \dfrac{\left(x^2-25^2\right)+\left(75^2-x^2\right)}{2} = 2500$.

So the maximum possible area, as desired, is $= \frac{3}{4}(2500) =$ **1875**.

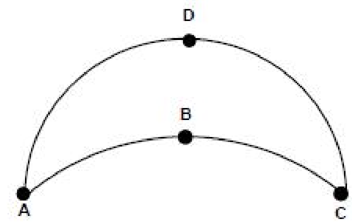
26. The polynomial $P(x) = a_0 + a_1x + a_2x^2 + .. + 10x^9$ has the property that $P\left(\frac{1}{k}\right) = \frac{1}{k}$ for k = 1, 2, 3, .., 9. Find $P\left(\frac{1}{10}\right)$.

Let $Q(x) = x.\, P\left(\frac{1}{x}\right) - 1 = 10(x - a)(x - 1)(x - 2)..(x - 9)$. As P is a polynomial of degree 9 so Q would be of degree 10 with a, 1, 2, 3, ..9 as zeroes. Now for x = 0, we have $a = -\frac{1}{10!}$.

Using above polynomial and value of 'a' we have $\boxed{\mathbf{P\left(\frac{1}{10}\right) = 10! + \frac{1}{5}}}$.

27. How many ordered triplets (a, b, c) of positive odd integers satisfy a + b + c = 23?
Let a = 2x + 1, b = 2y + 1, c = 2z + 1. So the given equation reduces to x + y + z = 10 where x, y, z are non-negative integers. Total number of solutions = $^{12}C_2$ = **66**.

28. The figure ABCD on the right is bounded by a semicircle ADC and a quarter-circle ABC. Given that shortest distance between A and C = 18 units. What is the area of region bounded by this figure?



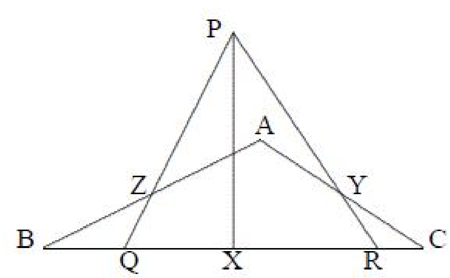
Radius of semicircle ADC = 9 and that of quarter-circle ABC = $9\sqrt{2}$. So required area = Area of semicircle – Area of quarter-circle + Area of triangular region in the quarter-circle = $\frac{\pi 9^2}{2} - \frac{\pi(9\sqrt{2})^2}{4} + \frac{1}{2}(9\sqrt{2})^2$ = **81 square units**.

29. A palindrome is a number which reads same forward and backward, e.g. 121 is a three digit palindrome number. What is the sum of all three digit palindromes which are multiple of 13?
Let N = aba = 100a + 10b + a = 91a + 10(a + b). Now (a + b) must be divisible by 13. Only possibility for (a + b) is 13. So N can be 494, 585, 676, 767, 858, 949. Sum of all the values = 111(4 + 5 + 6 + 7 + 8 + 9) = **4329**.

30. Find the sum of all the digits in the decimal representations of all the positive integers less than 1000.
Consider all the three digit numbers 000, 001, 002, …, 999. At each place all of the 10 digits (0, 1, 2, .., 9) have been used 10×10 = 100 times. So, in all, every digit is used 3×100 = 300 times while writing all the numbers.
Hence the required sum is = 300(0 + 1 + 2 + .. + 9) = 300×45 = **13500**.

31. Consider the numbers 3, 8, 13… 103, 108. What is the smallest value of *n* such that every collection of *n* of these numbers will always contain a pair which sums to 121?
Make the 12 groups as shown: (3), (8), (13, 108), (18, 103), (23, 98), (28, 93), (33, 88), (38, 83), (43, 78), (48, 73), (53, 68), and (58, 63). We can take maximum one member from each of the 12 groups so that it doesn't contain a pair of numbers which sums to 121. And next number from any group will make a pair which sums to 121. Hence minimum $\boxed{\mathbf{13}}$ of the numbers need to be selected so as to have a pair with given sum.

32. In the diagram shown, X is the midpoint of BC, Y is the midpoint of AC and Z is the midpoint of AB. Also ∠ABC + ∠PQC = ∠ACB + ∠PRB = 90°. Find ∠PXR.



∠QPR = 180° - ∠PQC - ∠PRB = ∠ABC + ∠ACB. Because ZY||BX and BZ||XY ∴∠ABC = ∠XYZ = ∠CXY also ∠ACB = ∠XZY = ∠BXZ.
Now ∠BXZ + ∠ZXY + ∠CXY = ∠ABC + ∠ACB + ∠ZXY = ∠QPR + ∠ZXY = 180°.
Hence PYXZ is a cyclic quadrilateral and ∠ZPX = ∠XYZ = ∠ABC.
=> In triangle PQX, ∠QPX + ∠PQX = 90°. So ∠PXQ = $\boxed{\mathbf{\angle PXR = 90°}}$.

33. Let a, b, c, d be four real numbers such that

a + b + c + d = 8,

ab + ac + ad + bc + bd + cd = 12.

Find the greatest possible value of d.

$a^2 + b^2 + c^2 + d^2 = (a + b + c + d)^2 - 2(ab + ac + ad + bc + bd + cd) = 40$.

For d to be greatest, a, band c must be least and equal. So let a = b = c = x

=> $x = \frac{8-d}{3}$ and $3\left(\frac{8-d}{3}\right)^2 + d^2 = 40$

=> $d^2 - 4d - 14 = 0$ => $\boxed{d_{max} = 2 + 3\sqrt{2}}$

34. The ordered pair of four-digit numbers (2025; 3136) has the property that each number in the pair is a perfect square and each digit of the second number is 1 more than the corresponding digit of the first number. Find all ordered pairs of five-digit numbers with the same property.

If $(n^2; m^2)$ is an ordered pair of 5-digit numbers satisfying the desired property, then we must have

$$11111 = m^2 - n^2 = (m - n)(m + n)$$

The number 11111 has only two factorizations into a product of two factors:

11111 = 41 × 271 and 11111 = 1 × 11111. Checking the two factorizations we have

(n; m) = (115; 156)

(n; m) = (5550; 5551)

The second pair will fail since n > 1000 implies that n^2 must have at least 7 digits. We check the first: $(n^2; m^2)$ = (13225; 24336): This pair works. So the **only 5-digit pair is (13225; 24336)**.

35. Exactly one of the statements in this problem is true. The first statement in this problem is false. In fact, both the first and second statements in this problem are false. How many true statements are there in this problem?

Only first statement is true and other two are false. Hence only **one** true statement is there.

36. Given that *a* and *b* are digits from 1 to 9, what is the number of fractions of the form *a/b*, expressed in lowest terms, which are less than 1?

We just need to find the sum ∅(1) + ∅(2) + ∅(3) + ∅(4) + ∅(5) + ∅(6) + ∅(7) + ∅(8) + ∅(9) = 0 + 1 + 2 + 2 + 4 + 2 + 6 + 4 + 6 = **27**. ∅(n) is Euler's totient function which gives the number of co prime numbers to n which are less than n.

37. For a positive integer n let f(n) be the value of $\frac{4n+\sqrt{4n^2-1}}{\sqrt{2n+1}+\sqrt{2n-1}}$. Calculate

f(1) + f(2) + … + f(40)

$$f(n) = \frac{4n+\sqrt{4n^2-1}}{\sqrt{2n+1}+\sqrt{2n-1}} = \frac{\left(\sqrt{2n+1}\right)^2+\left(\sqrt{2n+1}\right)\left(\sqrt{2n-1}\right)+\left(\sqrt{2n-1}\right)^2}{\left(\sqrt{2n+1}\right)+\left(\sqrt{2n-1}\right)}$$

$$= \frac{\left(\sqrt{2n+1}\right)^3-\left(\sqrt{2n-1}\right)^3}{\left(\sqrt{2n+1}\right)^2-\left(\sqrt{2n-1}\right)^2} = \frac{\left(\sqrt{2n+1}\right)^3-\left(\sqrt{2n-1}\right)^3}{2}$$

So $f(1) + f(2) + f(3) + \ldots + f(40) = \frac{\left(\sqrt{2\times40+1}\right)^3-\left(\sqrt{2\times1-1}\right)^3}{2} = \frac{729-1}{2} = \boxed{364}$.

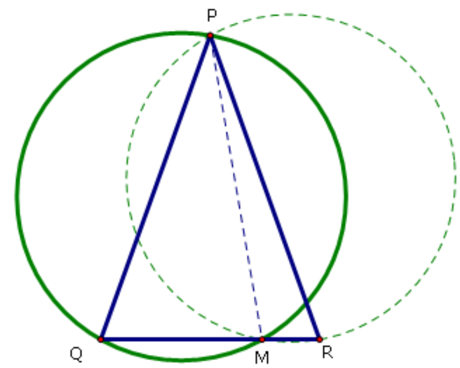
38. If N be the number of consecutive zeros at the end of the decimal representation of the expression 1! × 2! × 3! × 4! ×............× 99! × 100! Find the remainder when N is divided by 1000?

To find the number of zeroes we need to find highest power of 5 contained in the number. Given number can be written as $100^1 \times 99^2 \times 98^3 \times .. \times 2^{99} \times 1^{100}$. And to find highest power of 5 contained we need to look for multiples of 5 only. So N = (1 + 6 + 11 + .. + 96) + (1 + 26 + 51 + 76) = (10 × 97) + (2 × 77) = 970 + 154 = 1124 = **124** mod1000.

39. What are the dimensions of the greatest *n* × *n* square chessboard for which it is possible to arrange 121 coins on its cells so that the numbers of coins on any two adjacent cells (i.e. that share a side) differ by 1?

The parity of the number of coins in any two adjacent cells differs, so that at least one of any pair of adjacent cells contains at least one coin. This ensures that the number of cells cannot exceed 2 × 121 + 1 = 243 < 16^2, so that n ≤ 15. Since there are 121 coins, there must be an odd number of cells that contain an odd number of coins. We show that a **15 × 15** chessboard admits a suitable placement of coins. Begin by placing a single coin in every second cell so that each corner cell contains one coin. This uses up 113 coins. Now place two coins in each of four of the remaining 112 vacant cells. We have placed 113 + 8 = 121 coins in such a ways as to satisfy the condition.

40. Let PQR be an isosceles triangle with PQ = PR, and suppose that M is a point on the side QR with QR > QM > MR. Let QS and RT be diameters of the respective circumcircles of triangles PQM and PRM. What is the ratio QS : RT?



Common chord PM subtends equal angles at points Q and R in major segment and in the two circles respectively, because PQR is an isosceles triangle. So two circles are congruent and have equal diameter.

41. "You eat more than I do," said Tweedledee to Tweedledum.
"That is not true," said Tweedledum to Tweedledee.
"You are both wrong," said Alice to them both.
"You are right," said the White Rabbit to Alice.
How many of the four statements were true?

Only **one** statement is true. Either first or second one

42. The road from village *P* to village *Q* is divided into three parts. If the first section was 1.5 times as long and the second one was 2/3 as long as they are now, then the three parts would be all equal in length. What fraction of the total length of the road is the third section?

Let the total distance between two villages is x, then after alteration to the length of three sections each length is equal and thus is equal to x/3. So initial length of first section = 2x/9, second section = x/2, thus the initial length of third section = x − 2x/9 − x/2 = 5x/18. Hence the required fraction is **5/18**.

43. Four different digits are chosen and all possible positive four-digit numbers of distinct digits are constructed out of them. The sum of the four-digit numbers is 186 648. How many different sets of such four digits can be chosen?

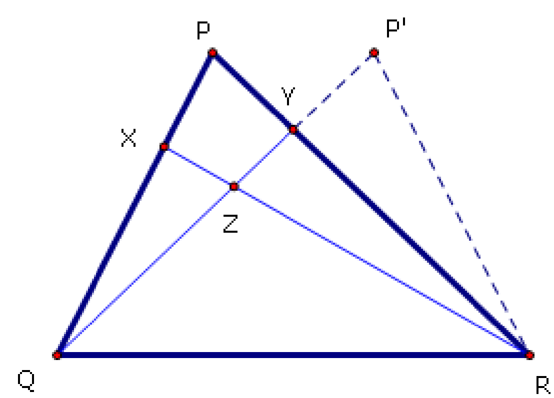
Let a, b, c, d be the four digits, then 3!(1111)(a + b + c + d) = 186 648.
That means a + b + c + d = 28. As a, b, c, d are different digits only following **two** sets can be chosen: {9, 8, 7, 4}, and {9, 8, 6, 5}.

44. If x = ±1 ±2 ±3 ±4 ±5 ±6 ±7 ±8 ±9 ±10. How many possible values can x take?

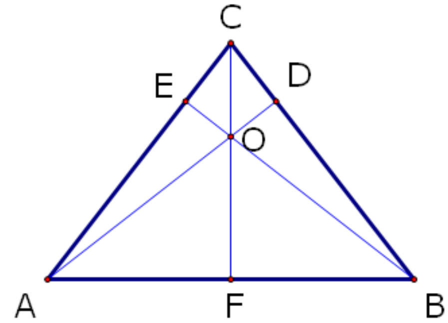
x can take all odd integers only from -55 to 55 i.e. **56** in all.

45. Points X and Y are on the sides PQ and PR of triangle PQR respectively. The segments QY and RX intersect at the point Z. Given that QY = RY, PQ = RZ and ∠QPR = 60°. Find ∠RZY.

Extend QY to P′ as shown, so that P′R = PQ. Now, ΔPYQ ≅ ΔP′YR. So ∠QPR = ∠RP′Z = 60°.
Also PQ = P′R = RZ. That means ∠RZY = ∠RP′Z = **60°**.



46. Let O, A, B, C be four points in a plane such that OA = OB = 15 and OC = 7. What is the maximum area of the triangle ABC?



For maximum area of triangle ABC, O has to be orthocenter as shown.
Triangle AOF and CBF are similar. So we have, by comparing the ratios of sides, $\frac{OF}{BF} = \frac{AF}{CF} = \frac{BF}{OF+7}$

=> $OF^2 + 7OF = BF^2 = AF^2 = 15^2 - OF^2$
=> $2OF^2 + 7OF - 225 = 0$
=> $(2OF + 25)(OF - 9) = 0$
=> OF = 9 and AF = 12
=> AB = 2AF = 24 and CF = 9 + 7 = 16
=> Area of triangle ABC = $\frac{1}{2} \times 24 \times 16$ = **192 sq. units**.

47. A particular month has 5 Tuesdays.
The first and the last day of the month are not Tuesday.
What day is the last day of the month?

Month must have 29, 30 or 31 days. 29 ≡ 1mod7 so 1st and 29th day would be having same day and that's not possible. If month has 30 days in that case also one of the first or last day has to be Tuesday to accommodate 5, hence not possible again. So the month has 31 days and 2nd & 30th day are Tuesday and the last day is **Wednesday**.

48. Find the minimum value of $\dfrac{\left(x+\frac{1}{x}\right)^6 - \left(x^6+\frac{1}{x^6}\right) - 2}{\left(x+\frac{1}{x}\right)^3 + \left(x^3+\frac{1}{x^3}\right)}$ for x > 0.

Let $x + \frac{1}{x} = y$ and $x^3 + \frac{1}{x^3} = z$, so the expression becomes $\frac{y^6 - z^2}{y^3 + z} = y^3 - z$ = 3y = $3\left(x + \frac{1}{x}\right)$.

By AM-GM inequality we know that $\left(x + \frac{1}{x}\right) \geq 2$. Hence minimum value of the expression is **6**.

49. What is the sum of the series: $2^2 + 4^2 + 6^2 + 10^2 + 16^2 + \dots + 754^2 + 1220^2$?

Required sum is $1220 \times (1220 + 754) - 2^2$ = **2408276**.
Observe that $2^2 + 2^2 + 4^2 + 6^2 + \dots + 754^2 + 1220^2 = 2(2 + 2) + 4^2 + 6^2 + \dots + 754^2 + 1220^2 = 4(2 + 4) + 6^2 + \dots + 754^2 + 1220^2 = 6(4 + 6) + \dots + 754^2 + 1220^2 = 1220(754 + 1220) = 2408280$.
Hence the required sum is: $2408280 - 2^2$ = **2408276**.

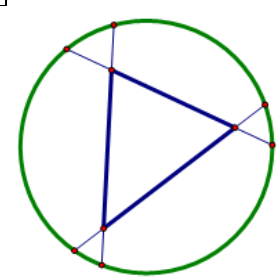
50. Determine *F(2010)* if for all real x and y, *F(x)F(y) – F(xy)* = x + y.

For x = y = 0, we have $F(0)[F(0) - 1] = 0$ => $F(0)$ = 0 or 1.
For x = 2010 and y = 0, we have $F(0)[F(2010) - 1] = 2010$
=> $F(0) \neq 0$ and $F(0) = 1$ hence $F(2010) = 2010 + 1$ = **2011**.

51. How many 4 digit number exist in which, when two digits are removed, 35 remains (e.g. 2315 and 3215 will be there in the list)?

3xxx + x3xx + xx3x = $(10^3 - 9^3) + 8(10^2 - 9^2) + 8 \times 9(10 - 9) = 271 + 152 + 72$ = **495**.

52. On a circle there are 10 points each of which is connected with each other with a straight line. How many triangles will be formed which lies completely inside the circle?



We need six points on the circle so that a triangle is formed with all the vertices lying inside the circle as shown below. So, in all, ${}^{10}C_6$ = **210** triangles will be formed.

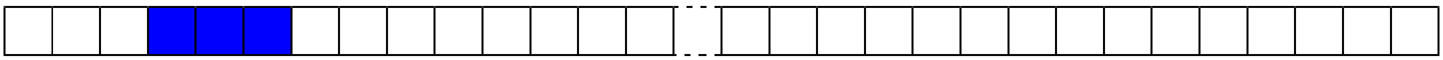
53. Let f(n) be the sum of the distinct positive prime divisors less than 50 for all positive integers n. For example: f(15) = 3 + 5 = 8 and f(61) = 0. Find the

remainder when f(1) + f(2) + … + f(99) is divided by 1000.

$f(1) + f(2) + \ldots + f(99) = 2\left\lfloor \frac{99}{2} \right\rfloor + 3\left\lfloor \frac{99}{3} \right\rfloor + 5\left\lfloor \frac{99}{5} \right\rfloor + \ldots + 47\left\lfloor \frac{99}{47} \right\rfloor = 1368 = \boxed{368} \text{ mod}1000.$

54. Two players A and B play a game moving alternately starting with A on a 1 × 100 grid of unpainted hundred unit squares. A has to paint three unpainted consecutive squares blue and B has to paint four unpainted consecutive squares red in their respective turns. The player who can not paint the squares in his turn loses. Who has the winning strategy?

A will paint his squares starting from fourth square from either edge of grid so as to have a reserve move whenever required. If in the course of time A cannot find three consecutive unpainted squares he will use the reserved squares and ensure a win. Thus **'A'** has a winning strategy.



55. Three men - Arthur, Bernard and Charles – with their wives – Ann, Barbara and Cynthia, not necessarily in order – make some purchases. When their shopping is finished each finds that the average cost in dollars of the articles he or she has purchased is equal to the number of his or her purchases. Arthur has bought 23 more articles than Barbara and Bernard has bought 11 more than Ann. Each husband has spent $63 more than his wife. What is the total amount spent by Charles and Cynthia?

As each husband has spent $63 more than her wife and also amount spent by each person is a perfect square. So, let the two amounts spent by a husband – wife pair be a^2 and b^2 respectively where a and b are number of articles purchased.

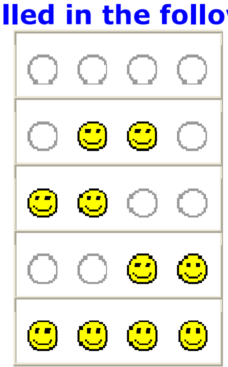
Solving $a^2 - b^2 = 63$ we get, (a, b) ≡ (32, 31), (12, 9) or (8, 1).

According to given information, we can make following table of husband-wife pairs:

Arthur	Bernard	Charles
32	12	8
Cynthia	Barbara	Ann
31	9	1

Hence the required total amount spent by Charles and Cynthia = $8^2 + 31^2 = 64 + 961 = \boxed{1025}$.

56. In TG's birthday bash people arrive in twos and want to sit next to their partner. How many ways can a row of 10 chairs be filled with couples or be left empty?
For instance, a row of 4 chairs can be filled in the following 5 ways:



For 1 chair only – there is 1 way, for 2 chairs – there are two ways, for 3 chairs – there are 3ways, for 4 chairs – there are 5 ways as shown. Number of ways is following the *Fibonacci pattern*. So the sequence is: 1, 2, 3, 5, 8, 13, 21, 34, 55, 89… Required number of ways is **89**.

57. How many sequences of 1's and 2's sum to 15?

Again *Fibonacci*. I think I've started loving it. Continuing with above pattern: 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987… Required number of ways is **987**.