153 PROBLEMS with SOLUTIONS

$153 = 1^3 + 5^3 + 3^3$

$153 = 1! + 2! + 3! + 4! + 5!$

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PROBLEMS

1. The number of persons who booked ticket for the New Year's concert is a perfect square. If 100 more persons booked ticket then the number of spectators would be a perfect square plus 1. If still 100 more persons booked ticket then the number of spectators would be again a perfect square. How many persons booked ticket for the concert?

2. If the last digits of the products $1\cdot 2$, $2\cdot 3$, $3\cdot 4$, ..., $n(n+1)$ are added, the result is 2010. How many products are used?

3. What is the remainder obtained when 2^{32} is divided by 641?

4. How many integers may be the measure, in degrees, of the angles of a regular polygon?

5. Several sets of prime numbers, such as {7, 83, 421, 659}, use each of the nine nonzero digits exactly once. What is the smallest possible sum such a set of primes could have?

6. How many ordered triplets (a, b, c) of non – zero real numbers have the property that each number is the product of the other two?

7. Rajat decided to tell the truth on Mondays, Thursdays and Saturdays, but lie on every other day. One day he says, "I will tell the truth tomorrow." What day of the week he made this statement?

8. For what smallest positive integral n, factorial of n is divisible by 414?

9. 2010 inhabitants of TG Land are divided into two groups: the Truth tellers – who always tell the truth and the Liars – who always tell a lie. Each person is exactly one of the following – a cricketer, a guitarist or a swimmer. Each inhabitant was asked the three questions: 1) Are you a cricketer? 2) Are you a guitarist? 3) Are you a swimmer? 1221 persons answered "yes" to the first question. 729 persons answered "yes" to second question and 660 persons answered "yes" to third question. How many "Liars" are present on the TG Land?

10. Isosceles triangle ABC has the property that, if D is a point on AC such that BD bisects angle ABC, then triangle ABC and BCD are similar. If BC has length of one unit, then what is the length of AB?

11. Vertices A, B and C of a parallelogram ABCD lie on a circle and D lies inside the circle such that line BD intersects the circle at point P. Given that $\angle APC = 75°$ and $\angle PAD = 19°$, what is the measure of $\angle PCD$?

12. Read the following 5 statements carefully:

(i) Statement (ii) is true.
(ii) At most one of the given five statements is true.
(iii) All of the given statements are true.
(iv) .
(v) .

The last two statements are printed in invisible ink. Which of the statements are true?

13. While adding all the page numbers of a book, I found the sum to be 1000. But then I realized that two page numbers (not necessarily consecutive) have not been counted. How many different pairs of two page numbers can be there?

14. How many positive integers are equal to 12 times the sum of their digits?

15. An 8 cm by 12 cm rectangle is folded along its long side so that two diagonally opposite corners coincide. What is the length of crease formed?

16. A point P inside an equilateral triangle, ABC is located at a distance of 3, 4 and 5 units respectively from A, B and C. What is the area of the triangle ABC?

17. Ten boxes each contain 9 balls. The balls in one box each weigh 0.9 kg; the rest all weigh 1 kg. In how many least number of weighing you can determine the box with the light balls?

18. A given circle has n chords. Each chord crosses every other chord but no three chords meet at the same point. How many regions are in the circle?

19. Find all prime numbers *p* for which *5p + 1* is a perfect square.

20. A programmer carelessly increased the tens digit by 1 for each multi-digit Fermat number in a lengthy list produced by a computer program. Fermat numbers are integers of the form: $N = 2^{2^n} + 1$ for integer n > 1. How many numbers on this new list are primes?

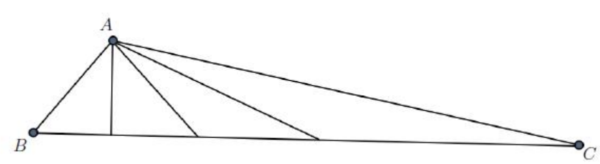
21. What is the greatest common divisor of the 2010 digit and 2005 digit numbers below?

$$\underbrace{33333...333}_{2010\ 3's} \quad \underbrace{7777...77}_{2005\ 7's}$$

22. Two players play a game on the board below as follows. Each person takes turns moving the letter **A** either downward at least one rectangle or to the left at least one rectangle (so each turn consists of moving either downward or to the left but not both). The first person to place the letter **A** on the rectangle marked with the letter **B** wins. How should the first player begin this game if we want to assure that he wins? Answer with the number given on the rectangle that he should move the letter **A** to.

1	2	3	4	5	6	7	8	9	**A**
									10
									11
B									12

23. In ΔABC (not drawn to scale), the altitude from *A*, the angle bisector of $\angle BAC$, and the median from *A* to the midpoint of *BC* divide $\angle BAC$ into four equal angles. What is the measure in degrees of angle $\angle BAC$?



24. Let $a_1, a_2, .. a_{2011}$ represents the arbitrary arrangement of the numbers 1, 2,..2011. Then what is the remainder when $(a_1 - 1)(a_2 - 2)..(a_{2011} - 2011)$ is divided by 2?

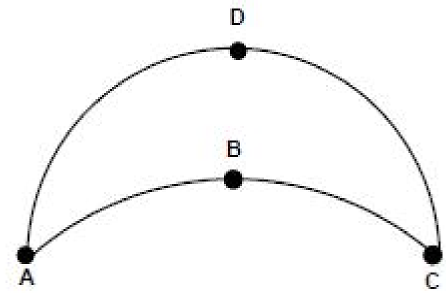
25. One side of a triangle has length 75. Of the other two sides, the length of one is double the length of the other. What is the maximum possible area for this triangle?

26. The polynomial $P(x) = a_0 + a_1x + a_2x^2 + \ldots + 10x^9$ has the property that $P\left(\frac{1}{k}\right) = \frac{1}{k}$ for k = 1, 2, 3, .., 9.

Find $P\left(\frac{1}{10}\right)$.

27. How many ordered triplets (a, b, c) of positive odd integers satisfy a + b + c = 23?

28. The figure ABCD on the right is bounded by a semicircle ADC and a quarter-circle ABC. Given that shortest distance between A and C = 18 units. What is the area of region bounded by this figure?

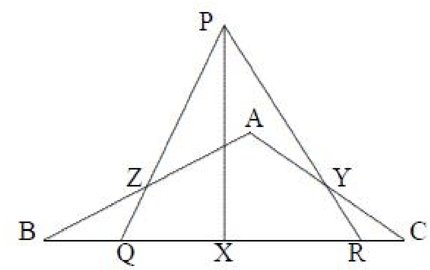


29. A palindrome is a number which reads same forward and backward, e.g. 121 is a three digit palindrome number. What is the sum of all three digit palindromes which are multiple of 13?

30. Find the sum of all the digits in the decimal representations of all the positive integers less than 1000.

31. Consider the numbers 3, 8, 13… 103, 108. What is the smallest value of *n* such that every collection of *n* of these numbers will always contain a pair which sums to 121?

32. In the diagram shown, X is the midpoint of BC, Y is the midpoint of AC and Z is the midpoint of AB. Also $\angle ABC + \angle PQC = \angle ACB + \angle PRB = 90°$. Find $\angle PXR$.



33. Let a, b, c, d be four real numbers such that

$$a + b + c + d = 8,$$
$$ab + ac + ad + bc + bd + cd = 12.$$

Find the greatest possible value of d.

34. The ordered pair of four-digit numbers (2025; 3136) has the property that each number in the pair is a perfect square and each digit of the second number is 1 more than the corresponding digit of the first number. Find all ordered pairs of five-digit numbers with the same property.

35. Exactly one of the statements in this problem is true. The first statement in this problem is false. In fact, both the first and second statements in this problem are false. How many true statements are there in this problem?

36. Given that *a* and *b* are digits from 1 to 9, what is the number of fractions of the form *a*/*b*, expressed in lowest terms, which are less than 1?

37. For a positive integer n let f(n) be the value of $\dfrac{4n + \sqrt{4n^2 - 1}}{\sqrt{2n+1} + \sqrt{2n-1}}$. Calculate

$$f(1) + f(2) + \ldots + f(40)$$

38. If N be the number of consecutive zeros at the end of the decimal representation of the expression 1! × 2! × 3! × 4! ×............× 99! × 100! Find the remainder when N is divided by 1000?

39. What are the dimensions of the greatest *n x n* square chessboard for which it is possible to arrange 121 coins on its cells so that the numbers of coins on any two adjacent cells (i.e. that share a side) differ by 1?

40. Let PQR be an isosceles triangle with PQ = PR, and suppose that M is a point on the side QR with QR > QM > MR. Let QS and RT be diameters of the respective circumcircles of triangles PQM and PRM. What is the ratio QS : RT?

41. "You eat more than I do," said Tweedledee to Tweedledum.
"That is not true," said Tweedledum to Tweedledee.
"You are both wrong," said Alice to them both.
"You are right," said the White Rabbit to Alice.
How many of the four statements were true?

42. The road from village *P* to village *Q* is divided into three parts. If the first section was 1.5 times as long and the second one was 2/3 as long as they are now, then the three parts would be all equal in length. What fraction of the total length of the road is the third section?

43. Four different digits are chosen, and all possible positive four-digit numbers of distinct digits are constructed out of them. The sum of the four-digit numbers is 186 648. How many different sets of such four digits can be chosen?

44. x = ±1 ±2 ±3 ±4 ±5 ±6 ±7 ±8 ±9 ±10. How many possible values can x take?

45. Points X and Y are on the sides PQ and PR of triangle PQR respectively. The segments QY and RX intersect at the point Z. Given that QY = RY, PQ = RZ and $\angle QPR = 60°$. Find $\angle RZY$.

46. Let O, A, B, C be four points in a plane such that OA = OB = 15 and OC = 7. What is the maximum area of the triangle ABC?

47. A particular month has 5 Tuesdays.
The first and the last day of the month are not Tuesday.
What day is the last day of the month?

48. Find the minimum value of $\dfrac{\left(x+\frac{1}{x}\right)^6-\left(x^6+\frac{1}{x^6}\right)-2}{\left(x+\frac{1}{x}\right)^3+\left(x^3+\frac{1}{x^3}\right)}$ for x > 0.

49. What is the sum of the series: $2^2 + 4^2 + 6^2 + 10^2 + 16^2 + ... + 754^2 + 1220^2$?

50. Determine *F(2010)* if for all real x and y, $F(x)F(y) - F(xy) = x + y$.

51. How many 4 digit number exist in which, when two digits are removed, 35 remains (e.g. 2315 and 3215 will be there in the list)?

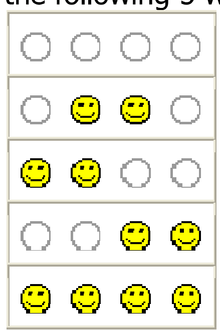
52. On a circle there are 10 points each of which is connected with each other with a straight line. How many triangles will be formed which lies completely inside the circle?

53. Let f(n) be the sum of the distinct positive prime divisors less than 50 for all positive integers n. For example: f(15) = 3 + 5 = 8 and f(61) = 0. Find the remainder when f(1) + f(2) + … + f(99) is divided by 1000.

54. Two players A and B play a game moving alternately starting with A on a 1×100 grid of unpainted hundred unit squares. A has to paint three unpainted consecutive squares blue and B has to paint four unpainted consecutive squares red in their respective turns. The player who can not paint the squares in his turn loses. Who has the winning strategy?

55. Three men - Arthur, Bernard and Charles – with their wives – Ann, Barbara and Cynthia, not necessarily in order – make some purchases. When their shopping is finished each finds that the average cost in dollars of the articles he or she has purchased is equal to the number of his or her purchases. Arthur has bought 23 more articles than Barbara, and Bernard has bought 11 more than Ann. Each husband has spent $63 more than his wife. What is the total amount spent by Charles and Cynthia?

56. In TG's birthday bash people arrive in twos and want to sit next to their partner. How many ways can a row of 10 chairs be filled with couples or be left empty?
For instance, a row of 4 chairs can be filled in the following 5 ways:



57. How many sequences of 1's and 2's sum to 15?

58. A closed bag contains 3 green hats and 2 red hats. Amar, Akbar, Anthony all close their eyes, take a hat, put it on, and close the bag. When they open their eyes, Amar looks at Akbar and Anthony, but can't deduce the color of his own hat. Akbar now tries to deduce his own hat's color but can't be certain. What color is Anthony's hat?

59. Find the sum of all remainders when $n^5 – 5n^3 + 4n$ is divided by 120 for all positive integers $n \geq 2010$.

60. The equation $x^2 + ax + (b + 2) = 0$ has real roots. What is the minimum value of $a^2 + b^2$?

61. There are 12 balls of equal size and shape, but one is either lighter or heavier than the other 11. For how many minimum number of times weighing required with ordinary beam balance to determine the faulty ball?

62. Sanjeev: I am thinking of a two digit number. Bet you can't guess it.
Kamal: Bet I can.
Sanjeev: Well, I'll only tell you the remainders of my number with anything from 1 to 10. How many questions do you think that you will have to ask?
Kamal: Hmmm! That depends on how lucky I am. But I'm not going to take chances. I am sure that I can guess your number with exactly _______ questions.
How many questions does Kamal tell Sanjeev he will ask?

63. What is the smallest possible difference between a square number and a prime number, if prime is greater than 3 and the square number is greater than prime?

64. 101 digits are chosen randomly and two numbers a, b are formed using all the digits exactly once. What is the probability that $a^4 = b$?

65. Let ABCD be a quadrilateral. The circumcircle of the triangle ABC intersects the sides CD and DA in the points P and Q respectively, while the circumcircle of CDA intersects the sides AB and BC in the points R and S. The straight lines BP and BQ intersect the straight line RS in the same points M and N respectively. If $\angle BQP = 90°$, find $\angle PMR$.

66. Kamal and Rajeev are playing the following game. They take turns writing down the digits of a six-digit number from left to right; Kamal writes the first digit, which must be nonzero, and repetition of digits is not permitted. Kamal wins the game if resulting six-digit number is divisible by 2, 3 or 5, and Rajeev wins otherwise.
Who has a winning strategy?

67. What is the least number of links you can cut in a chain of 21 links to be able to give someone all possible number of links up to 21?

68. Every blip is a blop. Half of all blops are blips, and half of all bleeps are blops. There are 30 bleeps and 20 blips. No bleep is a blip.
How many blops are neither blips nor bleeps?

69. Several weights are given, each of which is not heavier than 1 kg. It is known that they cannot be divided into two groups such that the weight of each group is greater than 1 kg. Find the maximum possible total weight of these weights.

70. Find the largest prime number p such that p^3 divided 2009! + 2010! + 2011!

71. How many integers less than 500 can be written as the sum of 2 positive integer cubes?

72. Three boys Ali, Bashar and Chirag are sitting around a round table in that order. Ali has a ball in his hand. Starting from Ali the boy having the ball passes it to either of the two boys. After 6 passes the ball goes back to Ali. How many different ways can the ball be passed?

73. There are 21 girls standing in a line. You have only nine chairs. In how many ways you can offer these chairs to nine select girls (one for each girl) such that number of standing girls between any two selected girls is odd?

74. In a group of people, there are 19 who like apples, 13 who like bananas, 17 who like cherries, and 4 who like dates. (A person can like more than 1 kind of fruit.) Each person who likes bananas also likes exactly one of apples and cherries. Each person who likes cherries also likes exactly one of bananas and dates. Find the minimum possible number of people in the group.

75. Let M and P be the points on sides AC and BC of ΔABC respectively such that AM : MC = 3 : 1 and BP : PC = 1 : 2. If Q is the intersection point of AP and BM and area of ΔBPQ is 1 square unit, find the area of ΔABC.

76. How many pairs of non-negative integers (x, y) satisfy $(xy - 7)^2 = x^2 + y^2$?