# 153 PROBLEMS with SOLUTIONS

$153 = 1^3 + 5^3 + 3^3$

$153 = 1! + 2! + 3! + 4! + 5!$

# PROBLEMS

1. The number of persons who booked ticket for the New Year's concert is a perfect square. If 100 more persons booked ticket then the number of spectators would be a perfect square plus 1. If still 100 more persons booked ticket then the number of spectators would be again a perfect square. How many persons booked ticket for the concert?

2. If the last digits of the products $1 \cdot 2$, $2 \cdot 3$, $3 \cdot 4$, ..., $n(n+1)$ are added, the result is 2010. How many products are used?

3. What is the remainder obtained when $2^{32}$ is divided by 641?

4. How many integers may be the measure, in degrees, of the angles of a regular polygon?

5. Several sets of prime numbers, such as {7, 83, 421, 659}, use each of the nine nonzero digits exactly once. What is the smallest possible sum such a set of primes could have?

6. How many ordered triplets (a, b, c) of non – zero real numbers have the property that each number is the product of the other two?

7. Rajat decided to tell the truth on Mondays, Thursdays and Saturdays, but lie on every other day. One day he says, "I will tell the truth tomorrow." What day of the week he made this statement?

8. For what smallest positive integral n, factorial of n is divisible by 414?

9. 2010 inhabitants of TG Land are divided into two groups: the Truth tellers – who always tell the truth and the Liars – who always tell a lie. Each person is exactly one of the following – a cricketer, a guitarist or a swimmer. Each inhabitant was asked the three questions: 1) Are you a cricketer? 2) Are you a guitarist? 3) Are you a swimmer? 1221 persons answered "yes" to the first question. 729 persons answered "yes" to second question and 660 persons answered "yes" to third question. How many "Liars" are present on the TG Land?

10. Isosceles triangle ABC has the property that, if D is a point on AC such that BD bisects angle ABC, then triangle ABC and BCD are similar. If BC has length of one unit, then what is the length of AB?

11. Vertices A, B and C of a parallelogram ABCD lie on a circle and D lies inside the circle such that line BD intersects the circle at point P. Given that $\angle APC = 75°$ and $\angle PAD = 19°$, what is the measure of $\angle PCD$?

12. Read the following 5 statements carefully:

(i) Statement (ii) is true.
(ii) At most one of the given five statements is true.
(iii) All of the given statements are true.
(iv) .
(v) .

The last two statements are printed in invisible ink. Which of the statements are true?

13. While adding all the page numbers of a book, I found the sum to be 1000. But then I realized that two page numbers (not necessarily consecutive) have not been counted. How many different pairs of two page numbers can be there?

14. How many positive integers are equal to 12 times the sum of their digits?

15. An 8 cm by 12 cm rectangle is folded along its long side so that two diagonally opposite corners coincide. What is the length of crease formed?

16. A point P inside an equilateral triangle, ABC is located at a distance of 3, 4 and 5 units respectively from A, B and C. What is the area of the triangle ABC?

17. Ten boxes each contain 9 balls. The balls in one box each weigh 0.9 kg; the rest all weigh 1 kg. In how many least number of weighing you can determine the box with the light balls?

18. A given circle has n chords. Each chord crosses every other chord but no three chords meet at the same point. How many regions are in the circle?

19. Find all prime numbers *p* for which $5p + 1$ is a perfect square.

20. A programmer carelessly increased the tens digit by 1 for each multi-digit Fermat number in a lengthy list produced by a computer program. Fermat numbers are integers of the form: $N = 2^{2^n} + 1$ for integer n > 1. How many numbers on this new list are primes?

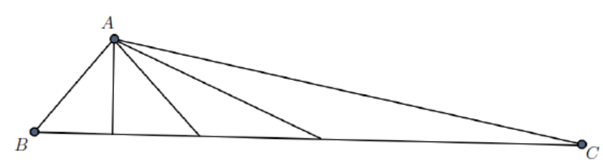
21. What is the greatest common divisor of the 2010 digit and 2005 digit numbers below?

$$\underbrace{33333...333}_{2010\ 3's} \quad \underbrace{7777...77}_{2005\ 7's}$$

22. Two players play a game on the board below as follows. Each person takes turns moving the letter **A** either downward at least one rectangle or to the left at least one rectangle (so each turn consists of moving either downward or to the left but not both). The first person to place the letter **A** on the rectangle marked with the letter **B** wins. How should the first player begin this game if we want to assure that he wins? Answer with the number given on the rectangle that he should move the letter **A** to.

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | **A** |
|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | 10 |
| | | | | | | | | | 11 |
| **B** | | | | | | | | | 12 |

23. In $\Delta ABC$ (not drawn to scale), the altitude from $A$, the angle bisector of $\angle BAC$, and the median from $A$ to the midpoint of $BC$ divide $\angle BAC$ into four equal angles. What is the measure in degrees of angle $\angle BAC$?




24. Let $a_1, a_2,..a_{2011}$ represents the arbitrary arrangement of the numbers 1, 2,..2011. Then what is the remainder when $(a_1 - 1)(a_2 - 2)..(a_{2011} - 2011)$ is divided by 2?

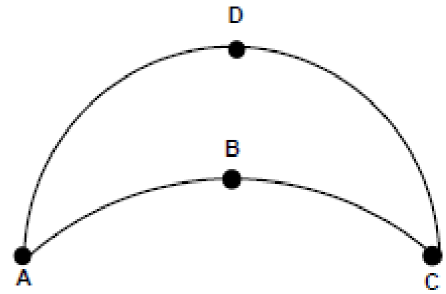
25. One side of a triangle has length 75. Of the other two sides, the length of one is double the length of the other. What is the maximum possible area for this triangle?

26. The polynomial $P(x) = a_0 + a_1x + a_2x^2 + \ \dots + 10x^9$ has the property that $P\left(\frac{1}{k}\right) = \frac{1}{k}$ for k = 1, 2, 3, .., 9.

Find $P\left(\frac{1}{10}\right)$.

27. How many ordered triplets (a, b, c) of positive odd integers satisfy a + b + c = 23?

28. The figure ABCD on the right is bounded by a semicircle ADC and a quarter-circle ABC. Given that shortest distance between A and C = 18 units. What is the area of region bounded by this figure?

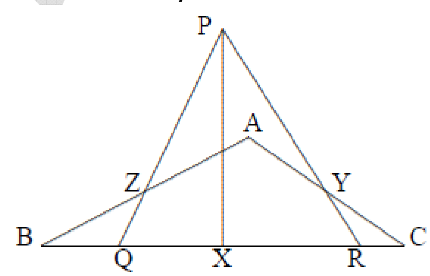



29. A palindrome is a number which reads same forward and backward, e.g. 121 is a three digit palindrome number. What is the sum of all three digit palindromes which are multiple of 13?

30. Find the sum of all the digits in the decimal representations of all the positive integers less than 1000.

31. Consider the numbers 3, 8, 13… 103, 108. What is the smallest value of *n* such that every collection of *n* of these numbers will always contain a pair which sums to 121?

32. In the diagram shown, X is the midpoint of BC, Y is the midpoint of AC and Z is the midpoint of AB. Also ∠ABC + ∠PQC = ∠ACB + ∠PRB = 90°. Find ∠PXR.




33. Let a, b, c, d be four real numbers such that

$$a + b + c + d = 8,$$
$$ab + ac + ad + bc + bd + cd = 12.$$

Find the greatest possible value of d.

34. The ordered pair of four-digit numbers (2025; 3136) has the property that each number in the pair is a perfect square and each digit of the second number is 1 more than the corresponding digit of the first number. Find all ordered pairs of five-digit numbers with the same property.

35. Exactly one of the statements in this problem is true. The first statement in this problem is false. In fact, both the first and second statements in this problem are false. How many true statements are there in this problem?

36. Given that *a* and *b* are digits from 1 to 9, what is the number of fractions of the form *a*/*b*, expressed in lowest terms, which are less than 1?

37. For a positive integer n let f(n) be the value of $\dfrac{4n + \sqrt{4n^2 - 1}}{\sqrt{2n+1} + \sqrt{2n-1}}$. Calculate

$$f(1) + f(2) + \dots + f(40)$$

38. If N be the number of consecutive zeros at the end of the decimal representation of the expression 1! × 2! × 3! × 4! ×...........× 99! × 100! Find the remainder when N is divided by 1000?

39. What are the dimensions of the greatest *n x n* square chessboard for which it is possible to arrange 121 coins on its cells so that the numbers of coins on any two adjacent cells (i.e. that share a side) differ by 1?

40. Let PQR be an isosceles triangle with PQ = PR, and suppose that M is a point on the side QR with QR > QM > MR. Let QS and RT be diameters of the respective circumcircles of triangles PQM and PRM. What is the ratio QS : RT?

41. "You eat more than I do," said Tweedledee to Tweedledum.
"That is not true," said Tweedledum to Tweedledee.
"You are both wrong," said Alice to them both.
"You are right," said the White Rabbit to Alice.
How many of the four statements were true?

42. The road from village *P* to village *Q* is divided into three parts. If the first section was 1.5 times as long and the second one was 2/3 as long as they are now, then the three parts would be all equal in length. What fraction of the total length of the road is the third section?

43. Four different digits are chosen, and all possible positive four-digit numbers of distinct digits are constructed out of them. The sum of the four-digit numbers is 186 648. How many different sets of such four digits can be chosen?

44. x = ±1 ±2 ±3 ±4 ±5 ±6 ±7 ±8 ±9 ±10. How many possible values can x take?

45. Points X and Y are on the sides PQ and PR of triangle PQR respectively. The segments QY and RX intersect at the point Z. Given that QY = RY, PQ = RZ and $\angle QPR = 60^\circ$. Find $\angle RZY$.

46. Let O, A, B, C be four points in a plane such that OA = OB = 15 and OC = 7. What is the maximum area of the triangle ABC?

47. A particular month has 5 Tuesdays.
The first and the last day of the month are not Tuesday.
What day is the last day of the month?

48. Find the minimum value of $\dfrac{\left(x+\frac{1}{x}\right)^6-\left(x^6+\frac{1}{x^6}\right)-2}{\left(x+\frac{1}{x}\right)^3+\left(x^3+\frac{1}{x^3}\right)}$ for x > 0.

49. What is the sum of the series: $2^2 + 4^2 + 6^2 + 10^2 + 16^2 + ... + 754^2 + 1220^2$?

50. Determine *F(2010)* if for all real x and y, *F(x)F(y) – F(xy) = x + y*.

51. How many 4 digit number exist in which, when two digits are removed, 35 remains (e.g. 2315 and 3215 will be there in the list)?

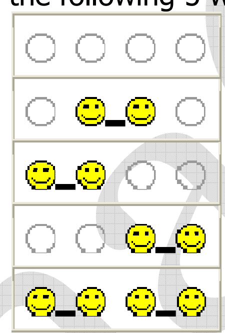
52. On a circle there are 10 points each of which is connected with each other with a straight line. How many triangles will be formed which lies completely inside the circle?

53. Let f(n) be the sum of the distinct positive prime divisors less than 50 for all positive integers n. For example: f(15) = 3 + 5 = 8 and f(61) = 0. Find the remainder when f(1) + f(2) + … + f(99) is divided by 1000.

54. Two players A and B play a game moving alternately starting with A on a $1 \times 100$ grid of unpainted hundred unit squares. A has to paint three unpainted consecutive squares blue and B has to paint four unpainted consecutive squares red in their respective turns. The player who can not paint the squares in his turn loses. Who has the winning strategy?

55. Three men - Arthur, Bernard and Charles – with their wives – Ann, Barbara and Cynthia, not necessarily in order – make some purchases. When their shopping is finished each finds that the average cost in dollars of the articles he or she has purchased is equal to the number of his or her purchases. Arthur has bought 23 more articles than Barbara, and Bernard has bought 11 more than Ann. Each husband has spent $63 more than his wife. What is the total amount spent by Charles and Cynthia?

56. In TG's birthday bash people arrive in twos and want to sit next to their partner. How many ways can a row of 10 chairs be filled with couples or be left empty?
For instance, a row of 4 chairs can be filled in the following 5 ways:



57. How many sequences of 1's and 2's sum to 15?

58. A closed bag contains 3 green hats and 2 red hats. Amar, Akbar, Anthony all close their eyes, take a hat, put it on, and close the bag. When they open their eyes, Amar looks at Akbar and Anthony, but can't deduce the color of his own hat. Akbar now tries to deduce his own hat's color but can't be certain. What color is Anthony's hat?

59. Find the sum of all remainders when $n^5 - 5n^3 + 4n$ is divided by 120 for all positive integers $n \geq 2010$.

60. The equation $x^2 + ax + (b + 2) = 0$ has real roots. What is the minimum value of $a^2 + b^2$?

61. There are 12 balls of equal size and shape, but one is either lighter or heavier than the other 11. For how many minimum number of times weighing required with ordinary beam balance to determine the faulty ball?

62. Sanjeev: I am thinking of a two digit number. Bet you can't guess it.
Kamal: Bet I can.
Sanjeev: Well, I'll only tell you the remainders of my number with anything from 1 to 10. How many questions do you think that you will have to ask?
Kamal: Hmmm! That depends on how lucky I am. But I'm not going to take chances. I am sure that I can guess your number with exactly _______ questions.
How many questions does Kamal tell Sanjeev he will ask?

63. What is the smallest possible difference between a square number and a prime number, if prime is greater than 3 and the square number is greater than prime?

64. 101 digits are chosen randomly and two numbers a, b are formed using all the digits exactly once. What is the probability that $a^4 = b$?

65. Let ABCD be a quadrilateral. The circumcircle of the triangle ABC intersects the sides CD and DA in the points P and Q respectively, while the circumcircle of CDA intersects the sides AB and BC in the points R and S. The straight lines BP and BQ intersect the straight line RS in the same points M and N respectively. If $\angle BQP = 90°$, find $\angle PMR$.

66. Kamal and Rajeev are playing the following game. They take turns writing down the digits of a six-digit number from left to right; Kamal writes the first digit, which must be nonzero, and repetition of digits is not permitted. Kamal wins the game if resulting six-digit number is divisible by 2, 3 or 5, and Rajeev wins otherwise.
Who has a winning strategy?

67. What is the least number of links you can cut in a chain of 21 links to be able to give someone all possible number of links up to 21?

68. Every blip is a blop. Half of all blops are blips, and half of all bleeps are blops. There are 30 bleeps and 20 blips. No bleep is a blip.
How many blops are neither blips nor bleeps?

69. Several weights are given, each of which is not heavier than 1 kg. It is known that they cannot be divided into two groups such that the weight of each group is greater than 1 kg. Find the maximum possible total weight of these weights.

70. Find the largest prime number $p$ such that $p^3$ divided 2009! + 2010! + 2011!

71. How many integers less than 500 can be written as the sum of 2 positive integer cubes?

72. Three boys Ali, Bashar and Chirag are sitting around a round table in that order. Ali has a ball in his hand. Starting from Ali the boy having the ball passes it to either of the two boys. After 6 passes the ball goes back to Ali. How many different ways can the ball be passed?

73. There are 21 girls standing in a line. You have only nine chairs. In how many ways you can offer these chairs to nine select girls (one for each girl) such that number of standing girls between any two selected girls is odd?

74. In a group of people, there are 19 who like apples, 13 who like bananas, 17 who like cherries, and 4 who like dates. (A person can like more than 1 kind of fruit.) Each person who likes bananas also likes exactly one of apples and cherries. Each person who likes cherries also likes exactly one of bananas and dates. Find the minimum possible number of people in the group.

75. Let M and P be the points on sides AC and BC of $\Delta ABC$ respectively such that AM : MC = 3 : 1 and BP : PC = 1 : 2. If Q is the intersection point of AP and BM and area of $\Delta BPQ$ is 1 square unit, find the area of $\Delta ABC$.

76. How many pairs of non-negative integers (x, y) satisfy $(xy - 7)^2 = x^2 + y^2$?

77. What is the 50th digit after decimal for:

$$\sqrt{\frac{2009 \times 2010 \times 2011 \times 2012 + 1}{4}}?$$

78. Year is 2051 and there is a strange game being played by 2051 inhabitants of TG Land. All 2051 inhabitants are standing in a circle. Now TG appears and randomly selects a person who shouts loudly IN, then person standing next clockwise say OUT, as must be the rule of the game, and get out of the circle. Again next person says IN and remain in his position and next says OUT and go out of circle. This process continues for a long time and in the end there is only one person remaining in the original circle. What is the position of the last survivor in the original circle, if first person selected by TG is numbered as 1 and numbers increases clockwise?

79. DaGny bought a rare earring set for \$700, sold it for \$800, bought it back for \$900 and sold it again for \$1000. How much profit did she make?

80. ABCD is a convex quadrilateral that is not parallelogram. P and Q are the midpoints of diagonals AC and BD respectively. PQ extended meets AB and CD at M and N respectively. Find the ratio of area($\Delta$ANB) : area($\Delta$CMD).

81. How many positive integers N are there such that $3 \times N$ is a three digit number and $4 \times N$ is a four digit number?

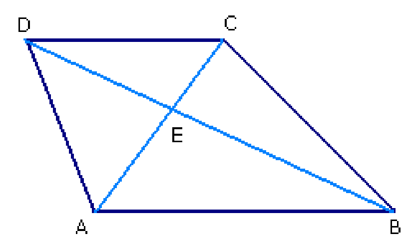
82. Lara is deciding whether to visit Kullu or Cherapunji for the holidays. She makes her decision by rolling a regular 6-sided die. If she gets a 1 or 2, she goes to Kullu. If she rolls a 3, 4, or 5, she goes to Cherapunji. If she rolls a 6, she rolls again. What is the probability that she goes to Cherapunji?

83. The numbers 201, 204, 209, 216, 225, ... are of the form $a_n = 200 + n^2$ where n = 1, 2, 3, 4, 5, .... For each n, let $D_n$ be the greatest common divisor of $a_n$ and $a_{n+1}$. What is the maximum value of $D_n$?

84. Messrs Baker, Cooper, Parson and Smith are a baker, a cooper, a parson and a smith. However, no one has the same name as his vocation. The cooper is not the namesake of Mr. Smith's vocation; the baker is neither Mr. Parson nor is he the namesake of Mr. Baker's vocation.

What is Mr. Baker's vocation?

85. In trapezium ABCD, AB||CD. If area($\Delta$ABE) = $\log_a 11$, area($\Delta$CDE) = $\log_{11} a$, and area($\Delta$ABC) = 11, find the area of ABCD.




86. Let P(x) be a polynomial such that:

$P(x) = x^{19} - 2011x^{18} + 2011x^{17} - ... - 2011x^2 + 2011x$. Calculate P(2010).

87. How many 9-digit numbers (in decimal system) divisible by 11 are there in which every digit occurs except zero?

88. There are four unit spheres inside a larger sphere, such that each of them touches the large sphere and the other three unit spheres. What is the radius of large sphere?

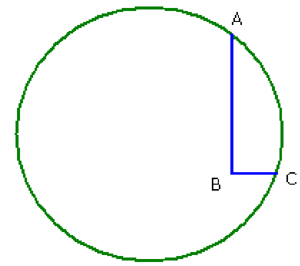
89. For the real numbers a, b and c, it is known that

$$\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}=1, \text{ and}$$

$$a + b + c = 1.$$

Find the value of the expression, M = $\frac{1}{1+a+ab}+\frac{1}{1+b+bc}+\frac{1}{1+c+ca}$.

90. In the circle shown, radius = $\sqrt{50}$, AB = 6, BC = 2, $\angle ABC$ = 90°. Find the distance from B to the centre of the circle.




91. Today is Friday. What day will it be after $4^{2010}$ days?

92. Solve the congruence cryptarithm $LIFE \equiv SIZE$ (mod $ELS$) in base 6 with E, L and S nonzero, all alphabets representing different numerals and Z > L > S.

Write the 6 letter-word denoting the digits 012345 as answer.

93. Find the sum $1+\frac{3}{2}+\frac{5}{4}+\frac{7}{8}+\ldots$up to infinity.

94. Two boats start at same instant from opposite ends of the river traveling across the water perpendicular to shores. Each travels at a constant but different speed. They pass at a point 720 meters from the nearest shore. Both boats remain at their slips for 15 minutes before starting back. On the return trip, they pass 400 meters from the other shore. Find the width of the river.

95. Larry, Curly, and Moe had an unusual combination of ages. The sum of any two of the three ages was the reverse of the third age (*e.g.*, 16 + 52 = 68, the reverse of 86). All were under 100 years old. What was the sum of the ages?

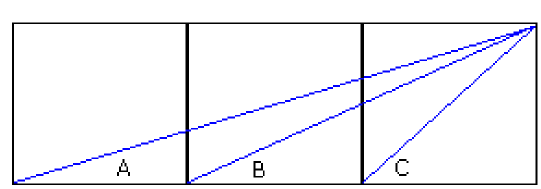
96. Find the sum of all four-digit numbers N whose sum of digits is equal to 2010 – N.

97. DaGny has 11 different colors of fingernail polish. Find the number of ways she can paint the five fingernails on her left hand by using at least three colors such that no two consecutive finger nails have same color. Also she is to apply only one color at one fingernail which is quite unusual for her.

98. A box contains 300 matches. Kamal and Sandeep take turns removing no more than half the matches in the box. The player who cannot move loses. What should be Kamal's first move to ensure his win if he is starting the game?

99. Let N be an integer such that $2N^2$ has exactly 28 distinct positive divisors and $3N^2$ has exactly 24 distinct positive divisors. How many distinct positive divisors does $6N^2$ have?

100. Three unit squares are joined as shown. Find the measure of $\angle A + \angle B + \angle C$.




101. What is the 625th term of the series where each term is made up of even digits only?

2, 4, 6, 8, 20, 22, 24, 26, 28, 40, 42, ...

102. The houses in a street are spaced so that each house of one lane is directly opposite to a house of other lane. The houses are numbered 1, 2, 3, … and so on up one side, continuing the order back down the other side. Number 39 is opposite to 66. How many houses are there?

103. In a polygon, internal angles have the measures of 90° and 270° only. If there are 18 angles of measure 270°, then what is the number of angles with measure of 90°?

104. How many pair of positive integers (a, b) are there such that their LCM is 2012?

105. What is the sum of all natural numbers which are less than 2012 and co-prime to it?

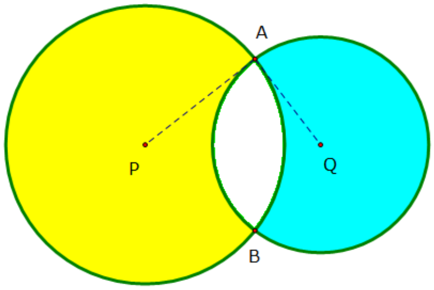
106. How many positive integers N satisfy: (i) $N < 1000$ and (ii) $N^2 - N$ is divisible by 1000?

107. 25 is a square number and can be written as average of two different square numbers i.e. 1 and 49. How many other square numbers from 1 to
625 inclusive can be written as average of two different square numbers?

108. I can break a block of 7 kg in smaller blocks of integral weights in four ways i.e. {1, 2, 4}, {1, 2, 2, 2}, {1, 1, 1, 4}, {1, 1, 1, 1, 1, 1, 1} such that I can measure each weight from 1 kg to 7 kg in exactly one way in either case.
For example, using 1st case this is the only possible combination of weights to measure 1 kg to 7 kg: 1 = 1, 2 = 2, 3 = 1 + 2, 4 = 4, 5 = 1 + 4, 6 = 2 + 4, and 7 = 1 + 2 + 4. So find the number of ways a block of 14 kg can be broken under similar conditions e.g. {1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1} is a valid case but {1, 2, 3, 4, 4} is not.

109. I am twice as old as you were when I was as old as you are. What is the ratio of ages of mine and yours?

110. Circles with centers P and Q have radii 20 and 15 cm respectively and intersect at two points A, B such that $\angle PAQ = 90°$. What is the difference in the area of two shaded regions?




111. What is the largest integer that is a divisor of

$$(n + 1)(n + 3)(n + 5)(n + 7)(n + 9)$$

for all positive even integers n?

112. For how many ordered pairs of positive integers (x, y), $xy/(x+y) = 9$?

113. Amu, Bebe, Chanda and Dori played with a deck of 52 cards. In one game, Dori was dealing out the cards one by one to the players, starting with Amu, followed by Bebe, Chanda and Dori in this order, when suddenly some of the cards she had not dealt out yet slipped out of her hands and fell on the floor. The girls noticed that the number of cards on the floor was 2/3 of the number of cards Amu had already got, and the

number of cards that Chanda had got was 2/3 of those in the remaining part of the deck in Dori's hand that she had not dealt out yet. How many cards had Dori dealt out altogether?

114. In a city, 2/3 of the men and 3/5 of the women are married. (Everyone has one spouse and the spouses live in the same city.) What fraction of the inhabitants of the city is married?

115. The sum of all interior angles of eight polygons is 3240°. What is the total number of sides of polygons?

116. Consider a triangle ABC with BC = 3. Choose a point D on BC such that BD = 2. Find the value of

$$AB^2 + 2AC^2 - 3AD^2.$$

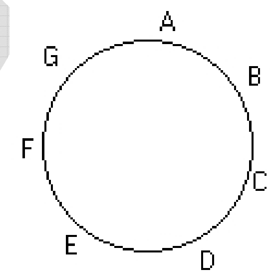
117. Determine the number of divisors of $2012^8$ that are less than $2012^4$.

118. How many numbers in the following sequence are prime numbers?

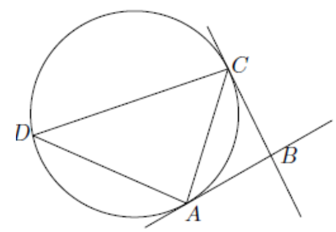
{1, 101, 10101, 1010101, 101010101, ……}

119. Find all triples of natural numbers (a, b, c) such that a, b and c are in geometric progression, and a + b + c = 111.

120. What is the smallest integer n for which $\sqrt{n} - \sqrt{n-1} < 0.01$?




121. Seven people, A, B, C, D, E, F and G can sit down for a meal at a round table as shown. Each person has two neighbours at the table: for example, A's neighbours are B and G. There are other ways in which the people can be seated round the table. Last month they dined together on a number of occasions, and no two of the people were neighbours more than once. How many meals could they have had together during the month?




122. Points A, D and C lie on the circumference of a circle. The tangents to the circle at points A and C meet at the point B. If ∠DAC = 83° and ∠DCA = 54°. Find ∠ABC.

123. How many 4-digit numbers uses exactly three different digits?

124. The ratio of two six digit numbers abcabc and ababab is 55 : 54. Find the value of a + b + c.

125. Find the infinite sum:

$$\frac{1}{4}+\frac{1}{8}+\frac{2}{16}+\frac{3}{32}+\frac{5}{64}+\frac{8}{128}+\frac{13}{256}+\frac{21}{512}+\ldots$$

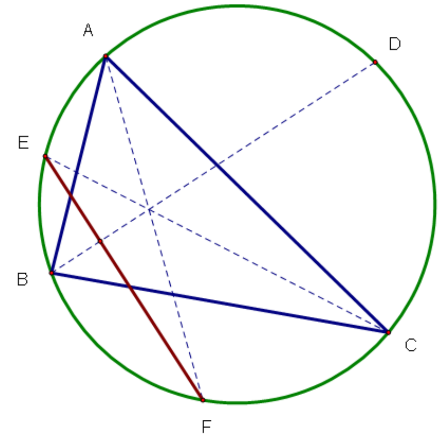
126. What is the probability of tossing a coin 6 times such that no two consecutive throws result in a head?

127. In how many ways 3 letters can be selected from 3 identical A's, 3 identical B's and 3 identical C's?

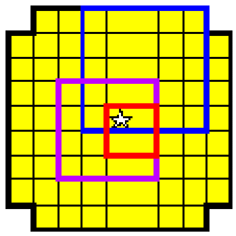
128. For how many pairs of positive integers (x, y) both $x^2 + 4y$ and $y^2 + 4x$ are perfect squares?

129. Jar X contains six liters of a 46% milk solution; Jar Y contains three liters of a 43% milk solution and Jar Z contains one liter of p% milk solution. q/r liters of solution from Jar Z is transferred to Jar X and remaining solution from Jar Z is transferred to Jar Y such that resulting two solutions both contain 50% milk solution. Also q and r are positive integers co-prime to each other. Find the value of p + q + r.

130. ABC is an isosceles right triangle inscribed in a circle such that $\angle B = 90°$. BD, CE and AF are angle bisectors of triangle ABC as shown. What is the measure of smaller angle of intersection of BD and EF?




131. In the diagram, three squares are shown, all containing the star. Altogether, how many squares containing the star can be found in the diagram?



132. Find all integers x, y, z (such that $x \le y \le z$) greater than 1 for which xy - 1 is divisible by z, yz - 1 is divisible by x, and zx - 1 is divisible by y.

133. In a rectangle ABCD, AB = 13 and BC = 8. PQ lies inside the rectangle such that BP = 11, DQ = 6 and AB||PQ, BP||DQ. Find the length of PQ.

134. What are the last two digits of the sum obtained by adding all the possible remainders of numbers of the form $2^n$, n being a non-negative integer, when divided by 100?

135. On planet LOGIKA, there live two kinds of inhabitants; black and white ones and they answer every question posed to them in a Yes or No. Black inhabitants of northern hemisphere always lie while white inhabitants of northern hemisphere always tell the truth. Also white inhabitants of southern hemisphere always lie while black inhabitants of southern hemisphere always tell the truth. On a dark night, there is an electricity failure and you meet an inhabitant without knowing your location on the planet. What single yes/no question can you ask the inhabitant to determine color of the inhabitant?

136. N is product of first 50 prime numbers. A is a factor of N and B is a factor of A. How many ordered pairs (A, B) exist?

137. For how many positive integers, N > 2, (N - 2)! + (N + 2)! is a perfect square?

138. Let P = $\frac{1}{1\times2}+\frac{1}{3\times4}+\frac{1}{5\times6}+\ldots+\frac{1}{2013\times2014}$ and

Q = $\frac{1}{1008\times2014}+\frac{1}{1009\times2013}+\frac{1}{1010\times2012}+\ldots.+\frac{1}{2014\times1008}$.

Find P/Q.

139. C and D are two points on a semicircle with AB as diameter such that AC – BC = 7 and AD – BD = 13. AD and BC intersect at P. Find the difference in area of triangles ACP and BDP.

140. TG *fashions* hold its annual sale on the eve of Pi-day ($14^{th}$ March) and offered a discount of 90% on all its apparels. But this month it is offering the usual 80% discount. How much percent more I need to pay now than that on the annual sale's eve for purchase of similar clothing?

141. In how many ways 1,000,000 can be expressed as sum of a square number and a prime number?

142. How many ordered triples of three positive integers (a, b, c) exist such that $a^3 + b^3 + c^3 = 2011$?

143. In a quadrilateral ABCD, sides AD and BC are parallel but not equal and sides AB = DC = x. The area of the quadrilateral is 676 $cm^2$. A circle with centre O and radius 13 cm is inscribed in the quadrilateral such that it is tangent to each of the four sides of the quadrilateral. Determine the length of x.

144. Kiran, Shashi and Rajni are Kiran's spouse, Shashi's sibling and Rajni's sister-in-law in no particular order. Also Kiran's spouse and Shashi's sibling are of same sex. Who among the three is a married male?

145. A and B start running from two opposite ends of a 1000m racing track. A and B travel with a speed of 8m/s and 5m/s respectively. How many times they meet, while running, in first 1000s after start?

146. According to death-will of Mr. Ranjan, all of this money was to be divided among his children in the following manner: `N to the first born plus 1/17 of what remains, `2N to the second born plus 1/17 of what then remains, `3N to the third born plus 1/17 of what then remains, and so on. When the distribution of the money was complete, each child received the same amount and no money was left over. Determine the number of children.

147. One number is removed from the set of integers from 1 to n. The average of the remaining numbers is 40.75. Which integer was removed?

148. What is the $2037^{th}$ positive integer that can be expressed as the sum of two or more consecutive positive integers? (The first three are 3 = 1+2, 5 = 2+3, and 6 = 1+2+3.)

149. Determine the number of ordered triplets (A, B, C) of sets which have the property that
(i) $A \cup B \cup C = \{1, 2, 3, \ldots, 1000\}$, and
(ii) $A \cap B \cap C = \varnothing$.

150. In a parallelogram ABCD, let M be the midpoint of the side AB and N the midpoint of BC. Let P be the intersection point of the lines MC and ND. Find the ratio of area of ▲s APB: BPC: CPD: DPA.

151. Kali-Jot is a game played by two players each of them having some number of marbles with her. One of the two players has to determine whether the number of marbles with other player is even or odd. A particular game of Kali-Jot has seven players and starts with players $P_1$ and $P_2$ on field and the other players $P_3$, $P_4$, $P_5$, $P_6$, $P_7$ waiting in a queue for their turn in order. After each game is played, the loser goes to the end of the queue; the winner adds 1 point to her score and stays on the field; and the player at the head of the queue comes on to contest the next point. Game continues until someone has scored 11 points. At that moment, it was found out that a total of 43 points have been scored by all seven players together. Who is the winner?

152. For positive real numbers; A, B, C, D such that A + B + C + D = 8, find the minimum possible value of $\frac{1}{A}+\frac{1}{B}+\frac{1}{C}+\frac{1}{D}$.

153. In two equilateral triangles ABC and BMN, $\angle ABM = 120°$. AN & CM intersects at O. Find $\angle MON$.

FOR ROUGH USE

# PROBLEMS with SOLUTIONS

**1. The number of persons who booked ticket for the New Year's concert is a perfect square. If 100 more persons booked ticket then the number of spectators would be a perfect square plus 1. If still 100 more persons booked ticket then the number of spectators would be again a perfect square. How many persons booked ticket for the concert?**

Let $a^2$ be the number of tickets booked initially which when increased by 200 becomes another perfect square, say $b^2$. So we know that $b^2 - a^2 = (b - a)(b + a) = 200 = (2)(100) = (4)(50) = (10)(20)$

Solving, we get $(a, b) \equiv (49, 51), (23, 27)$ or $(5, 15)$

But according to given conditions $a^2 + 100$ is just one more than a perfect square so only pair (49, 51) satisfies. Hence **number of persons who booked ticket for the concert = $49^2$ = 2401**.

**2. If the last digits of the products $1 \cdot 2$, $2 \cdot 3$, $3 \cdot 4$, …, $n(n+1)$ are added, the result is 2010. How many products are used?**

Unit digits of the numbers repeat after every 10 terms as 1, 2, 3, 4, 5, 6, 7, 8, 9, 0, … So unit digits of the product of two consecutive numbers as required can be found as:

| | | | | |
|---|---|---|---|---|
| 1 | × | 2 | = | 2 |
| 2 | × | 3 | = | 6 |
| 3 | × | 4 | = | 2 |
| 4 | × | 5 | = | 0 |
| 5 | × | 6 | = | 0 |
| 6 | × | 7 | = | 2 |
| 7 | × | 8 | = | 6 |
| 8 | × | 9 | = | 2 |
| 9 | × | 0 | = | 0 |
| 0 | × | 1 | = | 0 |

2, 6, 2, 0, 0, 2, 6, 2, 0, 0, … that repeats after every fifth term. Adding we get sum of unit digits of first five products as 2 + 6 + 2 + 0 + 0 = 10 which is same for next five numbers and so on. So to get a sum of 2010, 10 needs to be added 201 times or total 201 × 5 = 1005 products must have been used. But if we see the unit digit of 1004th and 1005th product are not contributing to the sum as they are zeroes only. Hence the **number of products used can be: 1003, 1004 or 1005**.

**3. What is the remainder obtained when $2^{32}$ is divided by 641?**

$2^8 \equiv 256 \text{ mod} 641$

$2^{16} = 2^8 2^8 \equiv 256^2 \text{ mod} 641 \equiv 65536 \text{ mod} 641 \equiv 154 \text{ mod} 641$

$2^{32} = 2^{16} 2^{16} \equiv 154^2 \text{ mod} 641 \equiv 23716 \text{ mod} 641 \equiv \mathbf{640} \text{ mod} 641$.

**4. How many integers may be the measure, in degrees, of the angles of a regular polygon?**

Each interior angle of a regular polygon of 'n' sides is given by $\left(180 - \frac{360}{n}\right)^\circ$.

For this angle to be an integer, 'n' must be a positive integral divisor of 360 greater than 2.

Number of positive integral divisors of 360 i.e. $(2^3 3^2 5) = 4 \times 3 \times 2 = 24$. Leaving 1 and 2 we have **22** different integers which can be the measure of each angle of a regular polygon.

**5. Several sets of prime numbers, such as {7, 83, 421, 659}, use each of the nine nonzero digits exactly once. What is the smallest possible sum such a set of primes could have?**

For smallest possible sum, we must try to make single digit and two digit prime numbers only. For two digit prime numbers, even digits (i.e. 2, 4, 6, 8) and 5 cannot appear on the units place so they must be at 10's place for sure (if used in two digit prime numbers). Also 1 is not a prime so must be used somewhere in two digit primes. But remember that 2 and 5 can be treated as single digit prime numbers.

So minimum we must have three two-digit prime numbers starting with, 4, 6 and 8 and remaining three numbers must be single digit primes for the smallest possible sum as desired. Hence the required sum is: 40 + 60 + 80 + 1 + 2 + 3 + 5 + 7 + 9 = **207**.

Required set of prime numbers can be {2, 3, 5, 41, 67, 89}, {2, 3, 5, 47, 61, 89}, {2, 5, 7, 43, 61, 89}

**6. How many ordered triplets (a, b, c) of non – zero real numbers have the property that each number is the product of the other two?**

There are **four** ordered triplets: (1, 1, 1), (-1, -1, 1), (-1, 1, -1), (1, -1, -1)

**7. Rajat decided to tell the truth on Mondays, Thursdays and Saturdays, but lie on every other day. One day he says, "I will tell the truth tomorrow." What day of the week he made this statement?**

His statement can't be true as in that case he'd have been speaking a truth for two consecutive days which is contrary to given information. Hence he is telling a lie and next day also he's going to lie. Two consecutive days on which he can tell a lie are Tuesday and Wednesday only. Hence he made the above statement on **Tuesday**.

**8. For what smallest positive integral n, factorial of n is divisible by 414?**

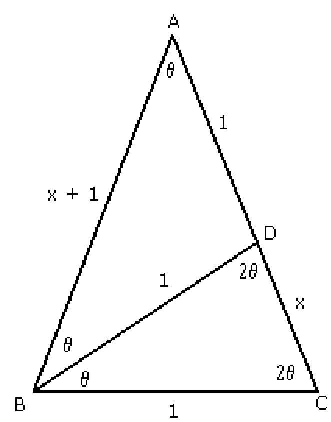
$414 = 2 \times 3^2 \times 23$. So n! must have atleast one 2, two 3's and one 23. Smallest possible n is **23**.

**9. 2010 inhabitants of TG Land are divided into two groups: the Truth tellers – who always tell the truth and the Liars – who always tell a lie. Each person is exactly one of the following – a cricketer, a guitarist or a swimmer. Each inhabitant was asked the three questions: 1) Are you a cricketer? 2) Are you a guitarist? 3) Are you a swimmer? 1221 persons answered "yes" to the first question. 729 persons answered "yes" to second question and 660 persons answered "yes" to third question. How many "Liars" are present on the TG Land?**

If each inhabitant must have spoken truthfully, then there should be 2010 'yes' in the response to the three questions. But due to each liar number of 'yes' will increase by one ( because when the person was telling truth, he must have said one 'yes' and two 'no' but if he will lie then he must say two 'yes' and one 'no').

Number of 'yes' received in response to three questions are: 1221 + 729 + 660 = 2610 which is 600 more than 2010. Hence **600** liars are present on the TG Land.

**10. Isosceles triangle ABC has the property that, if D is a point on AC such that BD bisects angle ABC, then triangle ABC and BCD are similar. If BC has length of one unit, then what is the length of AB?**




Given that triangle ABC is isosceles as shown in the diagram and also BD is bisector of $\angle ABC = 2\theta$.

So $\angle ABD = \angle CBD = \theta$ and $\angle ACD = 2\theta$ as shown below.

Also given that triangle ABC and BCD are similar so all the three angles of two triangles must be same. In triangle ABC two angles are equal to 2θ so must be in triangle BCD. Hence $\angle BDC = 2\theta$ and also $\angle BAC = \theta$ as shown in diagram above.

Hence triangle BCD is isosceles such that BD = BC = 1. And also triangle ADB is isosceles such that AD = BD = 1. Let CD = x, so that AB = x + 1.

Now equating the ratios of sides in similar triangles ABC and BCD, we have

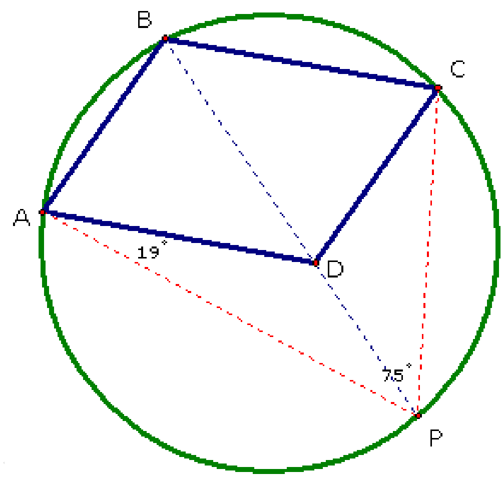
$$\frac{AB}{BC} = \frac{BC}{CD}$$

$$\frac{x+1}{1} = \frac{1}{x}$$

$\Rightarrow x^2 + x - 1 = 0$.

Because x is positive, $x = \frac{\sqrt{5}-1}{2}$ and $\boxed{AB = x + 1 = \frac{\sqrt{5}+1}{2}}$.

**11. Vertices A, B and C of a parallelogram ABCD lie on a circle and D lies inside the circle such that line BD intersects the circle at point P. Given that ∠APC = 75° and ∠PAD = 19°, what is the measure of ∠PCD?**




∠ABC = 180° - ∠APC = 105° (because ABCP is a cyclic quadrilateral)
Also ∠ADC = ∠ABC = 105° (opposite angles of a parallelogram are equal)
∠ADC = ∠ADB + ∠CDB = (∠PAD + ∠APD) + (∠PCD + ∠CPD)
105° = 19° + 75° + ∠PCD
=> **∠PCD = 11°**

**12. Read the following 5 statements carefully:**
**(i) Statement (ii) is true.**
**(ii) At most one of the given five statements is true.**
**(iii) All of the given statements are true.**
**(iv) .**
**(v) .**

**The last two statements are printed in invisible ink. Which of the statements are true?**
Both **statements (iv) and (v) are true**.

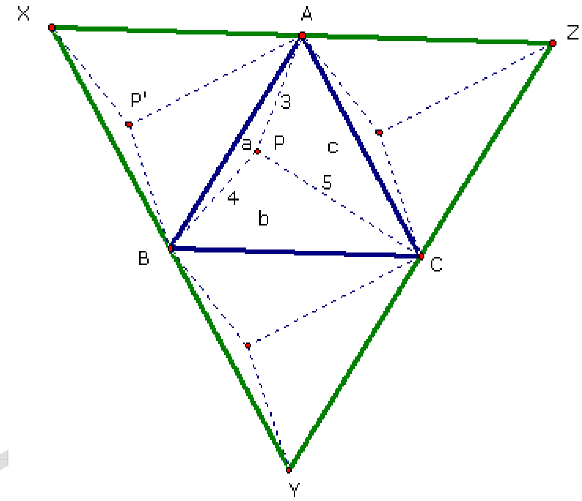
**13. While adding all the page numbers of a book, I found the sum to be 1000. But then I realized that two page numbers (not necessarily consecutive) have not been counted. How many different pairs of two page numbers can be there?**
Total number of pages can be either 45 as 1 + 2 + 3 +... + 45 = 1035 Or 46 as 1 + 2 + 3 +... + 46 = 1081.
So sum of missing page numbers is 35 or 81.
Now let a and b be the two page numbers. So in first case,
$a + b = 35$ and $a, b \in \{1, 2... 45\}$ i.e. total 17 cases - (1, 34), (2, 33)... (17, 18)
In second case, $a + b = 81$ and $a, b \in \{1, 2... 46\}$ i.e. total 6 cases – (35, 46), (36, 45)... (40, 41)
Hence in all, **23** different pairs of two page numbers can be there.

**14. How many positive integers are equal to 12 times the sum of their digits?**
Let the number have two digits so that $N = 10a + b = 12(a + b) \Rightarrow 2a + 11b = 0$ which is not possible.
If N is a three digit number, then $N = 100a + 10b + c = 12(a + b + c) \Rightarrow 88a = 2b + 11c$.
Only possibility is: a = 1, b = 0 and c = 8 i.e. N = **108**.
If N is a four digit number, then $N = 1000a + 100b + 10c + d = 12(a + b + c + d)$
$\Rightarrow 988a + 88b = 2c + 11d$. So RHS can never be greater than 117 and hence no four digit number can satisfy. Similarly any number with more than four digit will not satisfy the given property.
Hence only **One number** is equal to 12 times the sum of its digits.

**15. An 8 cm by 12 cm rectangle is folded along its long side so that two diagonally opposite corners coincide. What is the length of crease formed?**
This can be solved by making some right angle triangles and using Pythagoras theorem. A more beautiful solution is to observe that crease must form the perpendicular bisector of the diagonal. The length of the diagonal is $\sqrt{8^2 + 12^2} = 4\sqrt{13}$. If x is the length of crease then triangle formed by half the crease and half the diagonal is similar to the triangle formed by the diagonal and two sides of the rectangle.

So, $\frac{x/2}{2\sqrt{13}} = \frac{8}{12} \Rightarrow \boxed{x = \frac{8\sqrt{13}}{3}}$

**16. A point P inside an equilateral triangle, ABC is located at a distance of 3, 4 and 5 units respectively from A, B and C. What is the area of the triangle ABC?**




Let area of triangle ABC is divided into three triangles of area a, b and c as shown. Now rotate the triangle ABC from vertex B to XBA so that P is shifting to P'(say). Now PBP' is an equilateral triangle of side 4 units and APP' is a right angle triangle with side lengths (3, 4, 5).

So a + b = ar(PBP') + ar(APP') = $\frac{\sqrt{3}}{4}4^2$ + 6.

Similarly b + c = $\frac{\sqrt{3}}{4}5^2$ + 6, and c + a = $\frac{\sqrt{3}}{4}3^2$ + 6.

So area of triangle ABC = $\frac{(a+b)+(b+c)(c+a)}{2}$ = $\frac{\sqrt{3}}{4}5^2$ + 9.

Hence required area = $\frac{36+25\sqrt{3}}{4} \approx 19.8$ square units

**17. Ten boxes each contain 9 balls. The balls in one box each weigh 0.9 kg; the rest all weigh 1 kg. In how many least number of weighing you can determine the box with the light balls?**

Just number the boxes as #0, #1, #2… #9 and now take 0, 1, 2… 9 balls respectively from each box in that order. Now weigh all the 45 balls together in one weighing. Total weight will be (45 – a)kg. 'a' can easily be calculated from the weight obtained. Now box with lighter balls is '10a'. Hence only **one weighing** is sufficient.

**18. A given circle has n chords. Each chord crosses every other chord but no three chords meet at the same point. How many regions are in the circle?**

Number of region made by n intersecting chord is given by $^{n}C_2$ **+ n + 1**.

**19. Find all prime numbers *p* for which *5p + 1* is a perfect square.**

Let $5p + 1 = (x + 1)^2 \Rightarrow 5p = x(x + 2)$.

As LHS is product of two prime numbers so must be RHS. Also from RHS it is clear that difference between two prime numbers is 2 only. There are only **two** prime numbers i.e. 3 and 7 which are at a distance of two units from 5.

**20. A programmer carelessly increased the tens digit by 1 for each multi-digit Fermat number in a lengthy list produced by a computer program.**

**Fermat numbers are integers of the form: $N = 2^{2^n} + 1$ for integer n > 1. How many numbers on this new list are prime?**

By adding 1 to ten's digit we are just adding 10 to the number. So the given number becomes

$$N + 10 = 2^{2^n} + 11 \equiv (1 + 2) \bmod 3 \equiv 0 \bmod 3.$$

Hence all the new numbers are divisible by 3 and **none** is prime.

**21. What is the greatest common divisor of the 2010 digit and 2005 digit numbers below?**

$$\underbrace{33333...333}_{2010\ 3's} \quad \underbrace{7777...77}_{2005\ 7's}$$

Let A = 333..33 and B = 777..77, then A/3 and B/7 both contains the digit 1 only (2010 times and 2005 times respectively). As HCF(2010, 2005) = 5 that means HCF(A/3, B/7) = 11111.

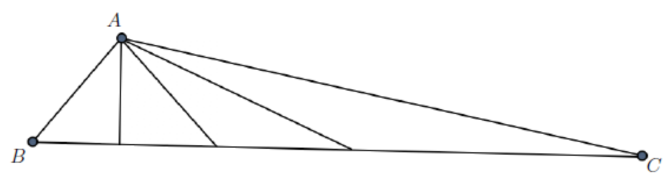
Now A is also divisible by 7 as 2010 = 6 × 335 (Remember a number formed by repeating same digit 6 times is divisible by 7). So HCF(A, B) = ***77777***.

**22. Two players play a game on the board below as follows. Each person takes turns moving the letter A either downward at least one rectangle or to the left at least one rectangle (so each turn consists of moving either downward or to the left but not both). The first person to place the letter A on the rectangle marked with the letter B wins. How should the first player begin this game if we want to assure that he wins? Answer with the number given on the rectangle that he should move the letter A to.**

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | **A** |
| | | | | | | | | | 10 |
| | | | | | | | | | 11 |
| **B** | | | | | | | | | 12 |

The first person should begin by moving *A* to the rectangle numbered 4. In fact, the first person can win by, on each turn, putting *A* on a rectangle along the diagonal from the rectangle labelled *B* to the rectangle numbered 4. The second player will have to move off this diagonal and then the first player can continue to move on the diagonal. Since the rectangle labelled with *B* is on the diagonal, the first person will eventually win (with this strategy). Observe that if first player does not put *A* on the rectangle numbered 4, then the second player can force a win by using the above strategy (putting *A* along the diagonal).

**23. In $\Delta ABC$ (not drawn to scale), the altitude from *A*, the angle bisector of $\angle BAC$, and the median from *A* to the midpoint of *BC* divide $\angle BAC$ into four equal angles. What is the measure in degrees of angle $\angle BAC$?**




Just apply Sine rule to find that **$\angle BAC = 90°$**.

**24. Let $a_1, a_2, .. a_{2011}$ represents the arbitrary arrangement of the numbers 1, 2,..2011. Then what is the remainder when $(a_1 - 1)(a_2 - 2)..(a_{2011} - 2011)$ is divided by 2?**

We have 1006 odd numbers and 1005 even numbers. So atleast one of the factors, which are multiplied, has to be the difference of two odd numbers and hence even. So the required remainder is **zero**.

**25. One side of a triangle has length 75. Of the other two sides, the length of one is double the length of the other. What is the maximum possible area for this triangle?**

Let other two sides be x and 2x, then using Heron's formula to find the area of triangle we have,

$$\Delta = \sqrt{\left(\frac{3x+75}{2}\right)\left(\frac{x+75}{2}\right)\left(\frac{-x+75}{2}\right)\left(\frac{3x-75}{2}\right)} = \frac{3}{4}\sqrt{\left(x^2-25^2\right)\left(75^2-x^2\right)}$$

Now area is maximum when $(x^2 - 25^2) = (75^2 - x^2) = \frac{\left(x^2-25^2\right)+\left(75^2-x^2\right)}{2} = 2500$.

So the maximum possible area, as desired, is $= \frac{3}{4}(2500) =$ **1875**.

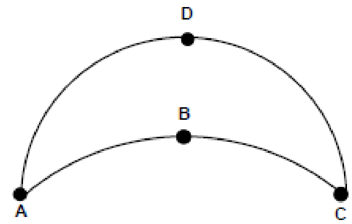
**26. The polynomial $P(x) = a_0 + a_1x + a_2x^2 + .. + 10x^9$ has the property that $P\left(\frac{1}{k}\right) = \frac{1}{k}$ for k = 1, 2, 3, .., 9. Find $P\left(\frac{1}{10}\right)$.**

Let $Q(x) = x.\ P\left(\frac{1}{x}\right) - 1 = 10(x - a)(x - 1)(x - 2)..(x - 9)$. As P is a polynomial of degree 9 so Q would be of degree 10 with a, 1, 2, 3, ..9 as zeroes. Now for x = 0, we have $a = -\frac{1}{10!}$.

Using above polynomial and value of 'a' we have $\boxed{\mathbf{P\left(\frac{1}{10}\right) = 10! + \frac{1}{5}}}$.

**27. How many ordered triplets (a, b, c) of positive odd integers satisfy a + b + c = 23?**
Let a = 2x + 1, b = 2y + 1, c = 2z + 1. So the given equation reduces to x + y + z = 10 where x, y, z are non-negative integers. Total number of solutions = $^{12}C_2$ = **66**.

**28. The figure ABCD on the right is bounded by a semicircle ADC and a quarter-circle ABC. Given that shortest distance between A and C = 18 units. What is the area of region bounded by this figure?**



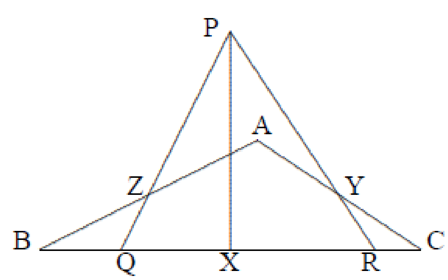

Radius of semicircle ADC = 9 and that of quarter-circle ABC = $9\sqrt{2}$. So required area = Area of semicircle – Area of quarter-circle + Area of triangular region in the quarter-circle = $\frac{\pi 9^2}{2} - \frac{\pi(9\sqrt{2})^2}{4} + \frac{1}{2}(9\sqrt{2})^2$ = **81 square units**.

**29. A palindrome is a number which reads same forward and backward, e.g. 121 is a three digit palindrome number. What is the sum of all three digit palindromes which are multiple of 13?**
Let N = aba = 100a + 10b + a = 91a + 10(a + b). Now (a + b) must be divisible by 13. Only possibility for (a + b) is 13. So N can be 494, 585, 676, 767, 858, 949. Sum of all the values = 111(4 + 5 + 6 + 7 + 8 + 9) = **4329**.

**30. Find the sum of all the digits in the decimal representations of all the positive integers less than 1000.**
Consider all the three digit numbers 000, 001, 002, …, 999. At each place all of the 10 digits (0, 1, 2, .., 9) have been used 10 × 10 = 100 times. So, in all, every digit is used 3 × 100 = 300 times while writing all the numbers.
Hence the required sum is = 300(0 + 1 + 2 + .. + 9) = 300 × 45 = **13500**.

**31. Consider the numbers 3, 8, 13… 103, 108. What is the smallest value of *n* such that every collection of *n* of these numbers will always contain a pair which sums to 121?**
Make the 12 groups as shown: (3), (8), (13, 108), (18, 103), (23, 98), (28, 93), (33, 88), (38, 83), (43, 78), (48, 73), (53, 68), and (58, 63). We can take maximum one member from each of the 12 groups so that it doesn't contain a pair of numbers which sums to 121. And next number from any group will make a pair which sums to 121. Hence minimum **13** of the numbers need to be selected so as to have a pair with given sum.

**32. In the diagram shown, X is the midpoint of BC, Y is the midpoint of AC and Z is the midpoint of AB. Also ∠ABC + ∠PQC = ∠ACB + ∠PRB = 90°. Find ∠PXR.**




∠QPR = 180° - ∠PQC - ∠PRB = ∠ABC + ∠ACB. Because ZY||BX and BZ||XY ∴∠ABC = ∠XYZ = ∠CXY also ∠ACB = ∠XZY = ∠BXZ.
Now ∠BXZ + ∠ZXY + ∠CXY = ∠ABC + ∠ACB + ∠ZXY = ∠QPR + ∠ZXY = 180°.
Hence PYXZ is a cyclic quadrilateral and ∠ZPX = ∠XYZ = ∠ABC.
=> In triangle PQX, ∠QPX + ∠PQX = 90°. So ∠PXQ = **∠PXR = 90°**.

**33. Let a, b, c, d be four real numbers such that**

**a + b + c + d = 8,**
**ab + ac + ad + bc + bd + cd = 12.**

**Find the greatest possible value of d.**

$a^2 + b^2 + c^2 + d^2 = (a + b + c + d)^2 - 2(ab + ac + ad + bc + bd + cd) = 40$.
For d to be greatest, a, band c must be least and equal. So let a = b = c = x

$\Rightarrow x = \frac{8-d}{3}$ and $3\left(\frac{8-d}{3}\right)^2 + d^2 = 40$

$\Rightarrow d^2 - 4d - 14 = 0 \Rightarrow \boxed{d_{max} = 2 + 3\sqrt{2}}$

**34. The ordered pair of four-digit numbers (2025; 3136) has the property that each number in the pair is a perfect square and each digit of the second number is 1 more than the corresponding digit of the first number. Find all ordered pairs of five-digit numbers with the same property.**

If $(n^2; m^2)$ is an ordered pair of 5-digit numbers satisfying the desired property, then we must have

$$11111 = m^2 - n^2 = (m - n)(m + n)$$

The number 11111 has only two factorizations into a product of two factors:
$11111 = 41 \times 271$ and $11111 = 1 \times 11111$. Checking the two factorizations we have

$$(n; m) = (115; 156)$$
$$(n; m) = (5550; 5551)$$

The second pair will fail since n > 1000 implies that $n^2$ must have at least 7 digits. We check the first: $(n^2; m^2) = (13225; 24336)$: This pair works. So the **only 5-digit pair is (13225; 24336)**.

**35. Exactly one of the statements in this problem is true. The first statement in this problem is false. In fact, both the first and second statements in this problem are false. How many true statements are there in this problem?**

Only first statement is true and other two are false. Hence only **one** true statement is there.

**36. Given that *a* and *b* are digits from 1 to 9, what is the number of fractions of the form *a*/*b*, expressed in lowest terms, which are less than 1?**

We just need to find the sum ∅(1) + ∅(2) + ∅(3) + ∅(4) + ∅(5) + ∅(6) + ∅(7) + ∅(8) + ∅(9) = 0 + 1 + 2 + 2 + 4 + 2 + 6 + 4 + 6 = **27**. ∅(n) is Euler's totient function which gives the number of co prime numbers to n which are less than n.

**37. For a positive integer n let f(n) be the value of $\frac{4n+\sqrt{4n^2-1}}{\sqrt{2n+1}+\sqrt{2n-1}}$. Calculate**

**f(1) + f(2) + … + f(40)**

$$f(n) = \frac{4n+\sqrt{4n^2-1}}{\sqrt{2n+1}+\sqrt{2n-1}} = \frac{\left(\sqrt{2n+1}\right)^2+\left(\sqrt{2n+1}\right)\left(\sqrt{2n-1}\right)+\left(\sqrt{2n-1}\right)^2}{\left(\sqrt{2n+1}\right)+\left(\sqrt{2n-1}\right)}$$

$$= \frac{\left(\sqrt{2n+1}\right)^3-\left(\sqrt{2n-1}\right)^3}{\left(\sqrt{2n+1}\right)^2-\left(\sqrt{2n-1}\right)^2} = \frac{\left(\sqrt{2n+1}\right)^3-\left(\sqrt{2n-1}\right)^3}{2}$$

So $f(1) + f(2) + f(3) + \ldots + f(40) = \frac{\left(\sqrt{2\times 40+1}\right)^3-\left(\sqrt{2\times 1-1}\right)^3}{2} = \frac{729-1}{2} = \boxed{364}$.

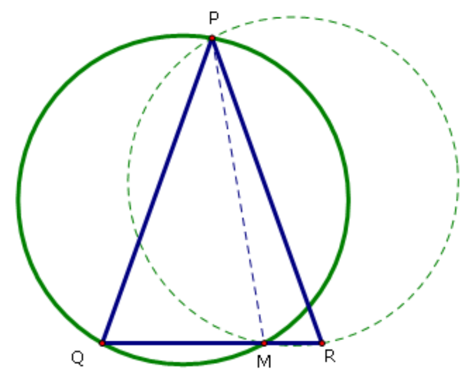
**38. If N be the number of consecutive zeros at the end of the decimal representation of the expression 1! × 2! × 3! × 4! ×…………× 99! × 100! Find the remainder when N is divided by 1000?**

To find the number of zeroes we need to find highest power of 5 contained in the number. Given number can be written as $100^1 \times 99^2 \times 98^3 \times .. \times 2^{99} \times 1^{100}$. And to find highest power of 5 contained we need to look for multiples of 5 only. So N = (1 + 6 + 11 + .. + 96) + (1 + 26 + 51 + 76) = (10 × 97) + (2 × 77) = 970 + 154 = 1124 = **124** mod1000.

**39. What are the dimensions of the greatest *n × n* square chessboard for which it is possible to arrange 121 coins on its cells so that the numbers of coins on any two adjacent cells (i.e. that share a side) differ by 1?**

The parity of the number of coins in any two adjacent cells differs, so that at least one of any pair of adjacent cells contains at least one coin. This ensures that the number of cells cannot exceed 2 × 121 + 1 = 243 < $16^2$, so that n ≤ 15. Since there are 121 coins, there must be an odd number of cells that contain an odd number of coins. We show that a **15 × 15** chessboard admits a suitable placement of coins. Begin by placing a single coin in every second cell so that each corner cell contains one coin. This uses up 113 coins. Now place two coins in each of four of the remaining 112 vacant cells. We have placed 113 + 8 = 121 coins in such a ways as to satisfy the condition.

**40. Let PQR be an isosceles triangle with PQ = PR, and suppose that M is a point on the side QR with QR > QM > MR. Let QS and RT be diameters of the respective circumcircles of triangles PQM and PRM. What is the ratio QS : RT?**




Common chord PM subtends equal angles at points Q and R in major segment and in the two circles respectively, because PQR is an isosceles triangle. So two circles are congruent and have equal diameter.

**41. "You eat more than I do," said Tweedledee to Tweedledum.**
**"That is not true," said Tweedledum to Tweedledee.**
**"You are both wrong," said Alice to them both.**
**"You are right," said the White Rabbit to Alice.**
**How many of the four statements were true?**

Only **one** statement is true. Either first or second one

**42. The road from village *P* to village *Q* is divided into three parts. If the first section was 1.5 times as long and the second one was 2/3 as long as they are now, then the three parts would be all equal in length. What fraction of the total length of the road is the third section?**

Let the total distance between two villages is x, then after alteration to the length of three sections each length is equal and thus is equal to x/3. So initial length of first section = 2x/9, second section = x/2, thus the initial length of third section = x – 2x/9 – x/2 = 5x/18. Hence the required fraction is **5/18**.

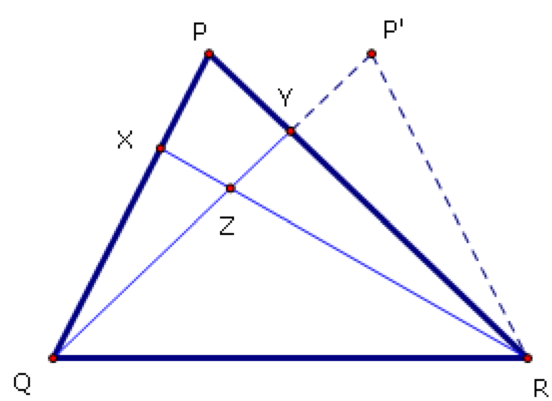
**43. Four different digits are chosen and all possible positive four-digit numbers of distinct digits are constructed out of them. The sum of the four-digit numbers is 186 648. How many different sets of such four digits can be chosen?**

Let a, b, c, d be the four digits, then 3!(1111)(a + b + c + d) = 186 648.
That means a + b + c + d = 28. As a, b, c, d are different digits only following **two** sets can be chosen: {9, 8, 7, 4}, and {9, 8, 6, 5}.

**44. If x = ±1 ±2 ±3 ±4 ±5 ±6 ±7 ±8 ±9 ±10. How many possible values can x take?**

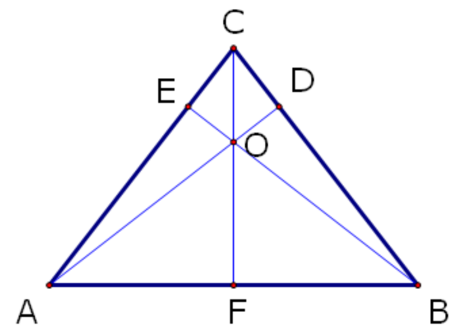
x can take all odd integers only from -55 to 55 i.e. **56** in all.

**45. Points X and Y are on the sides PQ and PR of triangle PQR respectively. The segments QY and RX intersect at the point Z. Given that QY = RY, PQ = RZ and ∠QPR = 60°. Find ∠RZY.**




Extend QY to P′ as shown, so that P′R = PQ. Now, ΔPYQ ≅ ΔP′YR. So ∠QPR = ∠RP′Z = 60°.
Also PQ = P′R = RZ. That means ∠RZY = ∠RP′Z = **60°**.

**46. Let O, A, B, C be four points in a plane such that OA = OB = 15 and OC = 7. What is the maximum area of the triangle ABC?**




For maximum area of triangle ABC, O has to be orthocenter as shown.
Triangle AOF and CBF are similar. So we have, by comparing the ratios of sides, $\frac{OF}{BF} = \frac{AF}{CF} = \frac{BF}{OF+7}$

=> $OF^2 + 7OF = BF^2 = AF^2 = 15^2 - OF^2$

=> $2OF^2 + 7OF - 225 = 0$

=> $(2OF + 25)(OF - 9) = 0$

=> OF = 9 and AF = 12

=> AB = 2AF = 24 and CF = 9 + 7 = 16

=> Area of triangle ABC = $\frac{1}{2} \times 24 \times 16$ = **192 sq. units**.

**47. A particular month has 5 Tuesdays.**
**The first and the last day of the month are not Tuesday.**
**What day is the last day of the month?**

Month must have 29, 30 or 31 days. 29 ≡ 1mod7 so 1st and 29th day would be having same day and that's not possible. If month has 30 days in that case also one of the first or last day has to be Tuesday to accommodate 5, hence not possible again. So the month has 31 days and 2nd & 30th day are Tuesday and the last day is **Wednesday**.

**48. Find the minimum value of $\dfrac{\left(x+\frac{1}{x}\right)^6 - \left(x^6+\frac{1}{x^6}\right) - 2}{\left(x+\frac{1}{x}\right)^3 + \left(x^3+\frac{1}{x^3}\right)}$ for x > 0.**

Let $x + \frac{1}{x} = y$ and $x^3 + \frac{1}{x^3} = z$, so the expression becomes $\frac{y^6 - z^2}{y^3 + z} = y^3 - z$ = 3y = $3\left(x + \frac{1}{x}\right)$.

By AM-GM inequality we know that $\left(x + \frac{1}{x}\right) \geq 2$. Hence minimum value of the expression is **6**.

**49. What is the sum of the series: $2^2 + 4^2 + 6^2 + 10^2 + 16^2 + ... + 754^2 + 1220^2$?**

Required sum is 1220 × (1220 + 754) - $2^2$ = **2408276**.

Observe that $2^2 + 2^2 + 4^2 + 6^2 + ... + 754^2 + 1220^2 = 2(2 + 2) + 4^2 + 6^2 + ... + 754^2 + 1220^2 = 4(2 + 4) + 6^2 + ... + 754^2 + 1220^2 = 6(4 + 6) + ...+ 754^2 + 1220^2 = 1220(754 + 1220) = 2408280$.

Hence the required sum is: 2408280 – $2^2$ = **2408276**.

**50. Determine *F(2010)* if for all real x and y, *F(x)F(y) – F(xy)* = *x* + *y*.**

For x = y = 0, we have *F(0)[F(0) – 1]* = 0 => *F(0)* = 0 or 1.

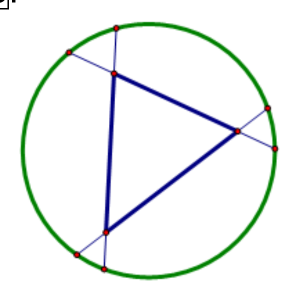
For x = 2010 and y = 0, we have *F(0)[F(2010) – 1]* = 2010

=> *F(0)* ≠ 0 and *F(0)* = 1 hence *F(2010)* = 2010 + 1 = **2011**.

**51. How many 4 digit number exist in which, when two digits are removed, 35 remains (e.g. 2315 and 3215 will be there in the list)?**

3xxx + x3xx + xx3x = $(10^3 - 9^3) + 8(10^2 - 9^2) + 8 \times 9(10 - 9)$ = 271 + 152 + 72 = **495**.

**52. On a circle there are 10 points each of which is connected with each other with a straight line. How many triangles will be formed which lies completely inside the circle?**



We need six points on the circle so that a triangle is formed with all the vertices lying inside the circle as shown below. So, in all, $^{10}C_6$ = **210** triangles will be formed.

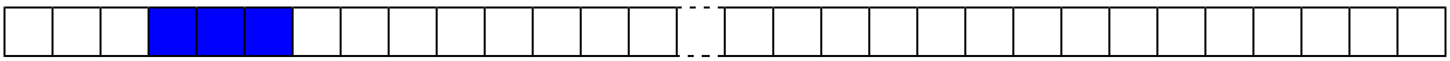
**53. Let f(n) be the sum of the distinct positive prime divisors less than 50 for all positive integers n. For example: f(15) = 3 + 5 = 8 and f(61) = 0. Find the**

**remainder when f(1) + f(2) + … + f(99) is divided by 1000.**

$f(1) + f(2) + \ldots + f(99) = 2\left\lfloor \frac{99}{2} \right\rfloor + 3\left\lfloor \frac{99}{3} \right\rfloor + 5\left\lfloor \frac{99}{5} \right\rfloor + \ldots + 47\left\lfloor \frac{99}{47} \right\rfloor = 1368 = \boxed{368} \text{ mod}1000.$

**54. Two players A and B play a game moving alternately starting with A on a 1 × 100 grid of unpainted hundred unit squares. A has to paint three unpainted consecutive squares blue and B has to paint four unpainted consecutive squares red in their respective turns. The player who can not paint the squares in his turn loses. Who has the winning strategy?**

A will paint his squares starting from fourth square from either edge of grid so as to have a reserve move whenever required. If in the course of time A cannot find three consecutive unpainted squares he will use the reserved squares and ensure a win. Thus **'A'** has a winning strategy.



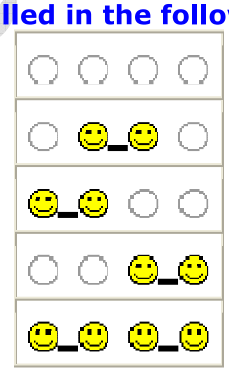
**55. Three men - Arthur, Bernard and Charles – with their wives – Ann, Barbara and Cynthia, not necessarily in order – make some purchases. When their shopping is finished each finds that the average cost in dollars of the articles he or she has purchased is equal to the number of his or her purchases. Arthur has bought 23 more articles than Barbara and Bernard has bought 11 more than Ann. Each husband has spent $63 more than his wife. What is the total amount spent by Charles and Cynthia?**

As each husband has spent $63 more than her wife and also amount spent by each person is a perfect square. So, let the two amounts spent by a husband – wife pair be $a^2$ and $b^2$ respectively where a and b are number of articles purchased.
Solving $a^2 - b^2 = 63$ we get, (a, b) ≡ (32, 31), (12, 9) or (8, 1).
According to given information, we can make following table of husband-wife pairs:

| | | |
|---|---|---|
| Arthur | Bernard | Charles |
| 32 | 12 | 8 |
| Cynthia | Barbara | Ann |
| 31 | 9 | 1 |

Hence the required total amount spent by Charles and Cynthia = $8^2 + 31^2 = 64 + 961 =$ **1025**.

**56. In TG's birthday bash people arrive in twos and want to sit next to their partner. How many ways can a row of 10 chairs be filled with couples or be left empty?**
**For instance, a row of 4 chairs can be filled in the following 5 ways:**



For 1 chair only – there is 1 way, for 2 chairs – there are two ways, for 3 chairs – there are 3ways, for 4 chairs – there are 5 ways as shown. Number of ways is following the *Fibonacci pattern*. So the sequence is: 1, 2, 3, 5, 8, 13, 21, 34, 55, 89… Required number of ways is **89**.

**57. How many sequences of 1's and 2's sum to 15?**

Again *Fibonacci*. I think I've started loving it. Continuing with above pattern: 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987… Required number of ways is **987**.

**58. A closed bag contains 3 green hats and 2 red hats. Amar, Akbar, Anthony all close their eyes, take a hat, put it on, and close the bag. When they open their eyes, Amar looks at Akbar and Anthony, but can't deduce the color of his own hat. Akbar now tries to deduce his own hat's color but can't be certain. What color is Anthony's hat?**

Since there were only 2 red hats in the bag, if Amar saw red hats on Akbar and Anthony, he would have known is own hat was green. Therefore Amar saw at least 1 green hat on either Akbar or Anthony. If Akbar saw a red hat on Anthony, he would have known that the green hat Amar saw was his own. Amar's and Akbar's silence allowed Anthony to deduce that his own hat's color must be **green**.

**59. Find the sum of all remainders when $n^5 - 5n^3 + 4n$ is divided by 120 for all positive integers $n \geq 2010$.**

$n^5 - 5n^3 + 4n = (n - 2)(n - 1)(n)(n + 1)(n + 2)$ i.e. product of 5 consecutive positive integers, hence is always divisible by 5! i.e. 120. So the required sum is **zero**.

**60. The equation $x^2 + ax + (b + 2) = 0$ has real roots. What is the minimum value of $a^2 + b^2$?**

$a^2 - 4(b + 2) = a^2 + b^2 - (b^2 + 4b + 8) \geq 0$.

So $a^2 + b^2 \geq b^2 + 4b + 8$

=> $a^2 + b^2 \geq (b + 2)^2 + 4$

=> $a^2 + b^2 \geq 4$.

Hence the required minimum value is **4**.

**61. There are 12 balls of equal size and shape, but one is either lighter or heavier than the other eleven. For how many minimum number of times weighing required with ordinary beam balance to determine the faulty ball?**

Begin by balancing 4 and 4. If they balance, one of the 4 remaining balls is different. Now choose 3 of the remaining 4 to balance against any 3 of the known good balls. If they don't balance, you've at least determined whether you're looking for a lighter or heavier ball, and it takes one balance to determine which of the 3 is faulty.

If the first 2 groups of 4 don't balance, it's a bit trickier. Let's suppose the left side is heavier, but remember that there could be a lighter ball on the right side. For the 2nd balance, replace 3 balls on the left (heavy) side with 3 balls from the remaining 4, and in addition, swap the 4th ball on the left side with any ball from the right side. If the scale now balances, you know that one of the 3 balls removed from the left side is heavier. If the left side is now lighter, one the 2 balls swapped is different. If the right side is still lighter, you know that one of the 3 balls on the right side that wasn't swapped is lighter. In any case, minimum **three** balances are required.

**Sanjeev:** **I am thinking of a two digit number. Bet you can't guess it.**
**Kamal:** **Bet I can.**
**Sanjeev:** **Well, I'll only tell you the remainders of my number with anything from 1 to 10. How many questions do you think that you will have to ask?**
**Kamal:** **Hmmm! That depends on how lucky I am. But I'm not going to take chances. I am sure that I can guess your number with exactly _______ questions.**

**62. How many questions does Kamal tell Sanjeev he will ask?**

Just **two** questions are sufficient. First Kamal will ask the remainder of number with 10 and in second question he'll ask for remainder of number with 9.

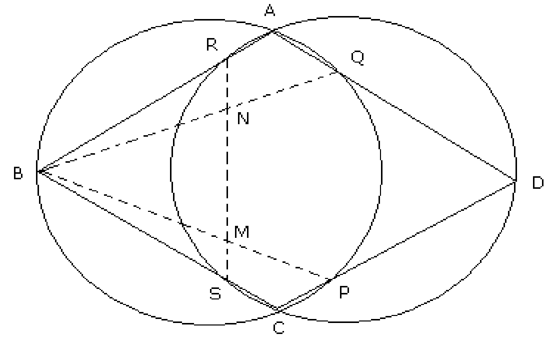
**63. What is the smallest possible difference between a square number and a prime number, if prime is greater than 3 and the square number is greater than prime?**

Least value is **2** because $3^2 - 7 = 2$. Say if least value is less than 2 i.e. 1, then $x^2 - 1$ should be prime which is not.

**64. 101 digits are chosen randomly and two numbers a, b are formed using all the digits exactly once. What is the probability that $a^4 = b$?**

If a has 'n' digits then $a^2$ can have 2n or 2n – 1 digits in turn $a^4$ or b will have 4n or 4n – 1 or 4n – 2 or 4n – 3 digits. So total digits used for a and b will be 5n or 5n – 1 or 5n – 2 or 5n – 3. But 101 is of the form 5n – 4. Hence the required probability is **zero**.

**65. Let ABCD be a quadrilateral. The circumcircle of the triangle ABC intersects the sides CD and DA in the points P and Q respectively, while the circumcircle of CDA intersects the sides AB and BC in the points R and S. The straight lines BP and BQ intersect the straight line RS in the same points M and N respectively. If ∠BQP = 90°, find ∠PMR.**




∠NQP = ∠BQP = ∠BAP = ∠BAC + ∠PAC = ∠RDC + ∠PBC = ∠MSB + ∠MBS = 180° - ∠BMS = 180° - ∠NMP.
=> ∠NQP + ∠NMP = 180°. Hence P, Q, M, N are concyclic and **∠PMR = 90°**.

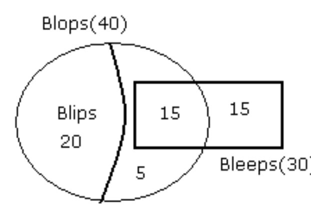
**66. Kamal and Rajeev are playing the following game. They take turns writing down the digits of a six-digit number from left to right; Kamal writes the first digit, which must be nonzero, and repetition of digits is not permitted. Kamal wins the game if resulting six-digit number is divisible by 2, 3 or 5, and Rajeev wins otherwise. Who has a winning strategy?**

Kamal will consume 3 and 9 in his first two moves so that Rajeev is left with at most 1 and 7 as his last move i.e. $6^{th}$ digit of the number. Now to win Kamal need to put his third move such that by putting this $5^{th}$ digit number becomes of the form 3k + 2 so that if Rajeev puts anything from 1 or 7, the number will be divisible by 3 or he'll need to put an even digit or 5. In either case **Kamal** has a winning strategy.

**67. What is the least number of links you can cut in a chain of 21 links to be able to give someone all possible number of links up to 21?**

**Two**. OOO C OOOOO C OOOOOOOOOOO (where Os are chained unbroken links, and the Cs are the unchained broken links). And equivalently: OOO C OOOOOO C OOOOOOOOOO

**68. Every blip is a blop. Half of all blops are blips, and half of all bleeps are blops. There are 30 bleeps and 20 blips. No bleep is a blip. How many blops are neither blips nor bleeps?**




They are **5** in number.

**69. Several weights are given, each of which is not heavier than 1 kg. It is known that they cannot be divided into two groups such that the weight of each group is greater than 1 kg. Find the maximum possible total weight of these weights.**

Three weights of 1 kg each will satisfy the condition. So the maximum possible total weight is **3 kg**.

**70. Find the largest prime number *p* such that $p^3$ divided 2009! + 2010! + 2011!**

2009! + 2010! + 2011! = 2009!(1 + 2010 + 2010 × 2011) = $2011^2$ × 2009! is divisible by $2011^2$ which is square of a prime number and all other prime numbers contained in 2009! are lesser than 2011. So we need to find the largest prime number which is contained thrice in 2009! i.e. largest prime number less than [2009/3] which is 661. Hence *p* is **661**.

**71. How many integers less than 500 can be written as the sum of 2 positive integer cubes?**

First observe that there are seven perfect cubes which are less than 500. Also, except $6^3 + 7^3$ and $7^3 + 7^3$, all sums of 2 perfect cubes are different numbers less than 500. So required answer is: ${}^8C_2$ – 2 = 28 – 2 = **26**.

**72. Three boys Ali, Bashar and Chirag are sitting around a round table in that order. Ali has a ball in his hand. Starting from Ali the boy having the ball passes it to either of the two boys. After 6 passes the ball goes back to Ali. How many different ways can the ball be passed?**

Table below shows the number of ways in which the ball can be passed to any three of the boys after every pass. For example; initially ball is with Ali so after $1^{st}$ pass it can go to Bashar in one way or Chirag in one way but it can't remain with Ali. Continuing on same thoughts after $2^{nd}$ pass, ball can be with Ali in two ways (passed either from Bashar or Chirag in one way each).

| Pass | Chirag | Ali | Bashar |
|---|---|---|---|
| $0^{th}$ | - | 1 | - |
| $1^{st}$ | 1 | 0 | 1 |
| $2^{nd}$ | 1 | 2 | 1 |
| $3^{rd}$ | 3 | 2 | 3 |
| $4^{th}$ | 5 | 6 | 5 |
| $5^{th}$ | 11 | 10 | 11 |
| $6^{th}$ | 21 | **22** | 21 |

Clearly the required answer is **22**.

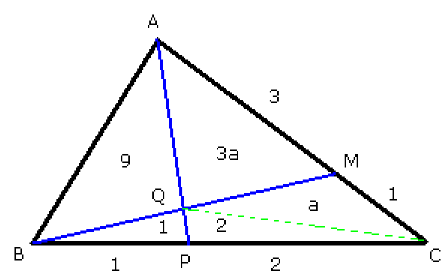
**73. There are 21 girls standing in a line. You have only nine chairs. In how many ways you can offer these chairs to nine select girls (one for each girl) such that number of standing girls between any two selected girls is odd?**

Observe that if we number the girls from 1 to 21, then the nine selected girls must be of same parity i.e. either all odd numbered or all even numbered. So required number of ways are $^{11}C_9 + {}^{10}C_9$ = **65**.

**74. In a group of people, there are 19 who like apples, 13 who like bananas, 17 who like cherries, and 4 who like dates. (A person can like more than 1 kind of fruit.) Each person who likes bananas also likes exactly one of apples and cherries. Each person who likes cherries also likes exactly one of bananas and dates. Find the minimum possible number of people in the group.**

As each of 17 person who likes cherries also like exactly one of bananas and dates, so 13 persons who like bananas all like cherries and do not like apples. Also all 4 persons who like dates also like cherries. Now out of 19 people who like apples, 4 can be common with date likers. So minimum number of people in the group is 17 + (19 - 4) = 17 + 15 = **32**.

**75. Let M and P be the points on sides AC and BC of ΔABC respectively such that AM : MC = 3 : 1 and BP : PC = 1 : 2. If Q is the intersection point of AP and BM and area of ΔBPQ is 1 square unit, find the area of ΔABC.**




Join QC. As BP : PC = 1 : 2, area of ΔQPC = 2. Let area of ΔCQM = a, then area of ΔAQM = 3a because AM : CM = 3 : 1. Also area of ΔAQB = 9 because of same reason. As area of ΔABP = 10 units so area of ΔABC = 3 × area of ΔABP = **30 units**.

**76. How many pairs of non-negative integers (x, y) satisfy $(xy - 7)^2 = x^2 + y^2$?**

$(xy - 7)^2 = x^2y^2 - 14xy + 49 = x^2 + y^2$

=> $x^2y^2 - 2(xy)(6) + 6^2 + 13 = x^2 + y^2 + 2xy$

=> $(xy - 6)^2 + 13 = (x + y)^2$

=> $(x + y + xy - 6)(x + y - xy + 6) = 13$

=> (x, y) = (0, 7), (7, 0), (3, 4) or (4, 3). So there are **four** such pairs.

**77. What is the $50^{th}$ digit after decimal for: $\sqrt{\frac{2009 \times 2010 \times 2011 \times 2012 + 1}{4}}$?**

$x(x + 1)(x + 2)(x + 3) + 1 = (x^2 + 3x + 1 - 1)(x^2 + 3x + 1 + 1) + 1 = (x^2 + 3x + 1)^2$

So $2009 \times 2010 \times 2011 \times 2012 + 1$ is an odd perfect square and the given expression is half of an odd integer which will give only one digit i.e. 5 after decimal. So $50^{th}$ digit after decimal is **zero**.

**78.Year is 2051 and there is a strange game being played by 2051 inhabitants of TG Land. All 2051 inhabitants are standing in a circle. Now TG appears and randomly selects a person who shouts loudly IN, then person standing next clockwise say OUT, as must be the rule of the game, and get out of the circle. Again next person says IN and remain in his position and next says OUT and go out of circle. This process continues for a long time and in the end there is only one person remaining in the original circle. What is the position of the last survivor in the original circle, if first person selected by TG is numbered as 1 and numbers increases clockwise?**

Check for smaller number of persons, when total number of persons is a power of 2, then the survivor is person1 other wise it increases to next odd number as 3, 5, 7, ... and so on as can be observed from below table:

| Total Persons | Survivor | | Total Persons | Survivor | | Total Persons | Survivor | | Total Persons | Survivor |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | | 2 | 1 | | 4 | 1 | | 8 | 1 |
| | | | 3 | 3 | | 5 | 3 | | 9 | 3 |
| | | | | | | 6 | 5 | | 10 | 5 |
| | | | | | | 7 | 7 | | 11 | 7 |
| | | | | | | | | | 12 | 9 |
| | | | | | | | | | 13 | 11 |
| | | | | | | | | | 14 | 13 |
| | | | | | | | | | 15 | 15 |

Also, if you observe from above table the binary representation of the numbers, you can notice if leading digit 1 of total persons if switched to unit's place then we get the last survivor.

So if total persons are 2051 which is 2048 + 1 + 1 + 1, the last survivor is **7**.

A better solution using recursion is that, if we have total number of persons as n, then let the survivor be S(n). Now total number of persons may be 2n or 2n + 1.

If we have total 2n persons, then in first round of game all even numbered n persons will be eliminated leaving n persons 1, 3, 5, ...., 2n – 1. So we can comfortably write, S(2n) = 2S(n) – 1.

Similarly it can be easily proved for 2n + 1 persons that S(2n + 1) = 2S(n) + 1.

Now because: $2051 = 2 \times 1025 + 1$
and $1025 = 2 \times 512 + 1$
and $512 = 2 \times 256$
and $256 = 2 \times 128$
and $128 = 2 \times 64$
and $64 = 2 \times 32$
and $32 = 2 \times 16$
and $16 = 2 \times 8$
and $8 = 2 \times 4$
and $4 = 2 \times 2$
and $2 = 2 \times 1$

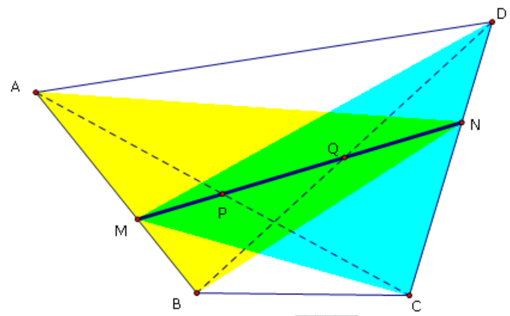
Now moving back, we know that S(1) = 1
so S(2) = 2S(1) – 1 = 1
so S(4) = 2S(2) – 1 = 1
so S(8) = 2S(4) – 1 = 1
so S(16) = 2S(8) – 1 = 1
so S(32) = 2S(16) – 1 = 1
so S(64) = 2S(32) – 1 = 1
so S(128) = 2S(64) – 1 = 1

so S(256) = 2S(128) – 1 = 1
so S(512) = 2S(256) – 1 = 1
so S(1025) = 2S(512) + 1 = 3
so S(2051) = 2S(1025) + 1 = **7**.

**79. DaGny bought a rare earring set for $700, sold it for $800, bought it back for $900 and sold it again for $1000. How much profit did she make?**

She made a profit of $100 in both transactions so total profit is **$200**.

**80. ABCD is a convex quadrilateral that is not parallelogram. P and Q are the midpoints of diagonals AC and BD respectively. PQ extended meets AB and CD at M and N respectively. Find the ratio of area(ΔANB) : area(ΔCMD).**




Let P and Q be the mid points of diagonals AC and BD as shown.
Area of triangle (APN) = Area of triangle (CPN)
Area of triangle (APM) = Area of triangle (CPM)
Area of triangle (BQN) = Area of triangle (DQN)
Area of triangle (BQM) = Area of triangle (DQM)
Adding all the four equations we have,
Area of triangle (ABN) = Area of triangle (CDM) as required. So the required ratio is **1 : 1**.

**81. How many positive integers N are there such that 3 × N is a three digit number and 4 × N is a four digit number?**

N varies from 250 to 333 inclusive, so total numbers of different values are: 333 – 249 = **84**.

**82. Lara is deciding whether to visit Kullu or Cherapunji for the holidays. She makes her decision by rolling a regular 6-sided die. If she gets a 1 or 2, she goes to Kullu. If she rolls a 3, 4, or 5, she goes to Cherapunji. If she rolls a 6, she rolls again. What is the probability that she goes to Cherapunji?**

The required probability is $= \frac{1}{2} + \left(\frac{1}{6}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{6}\right)^2\left(\frac{1}{2}\right) + \left(\frac{1}{6}\right)^3\left(\frac{1}{2}\right) + \ldots = \frac{\frac{1}{2}}{1 - \frac{1}{6}} = \boxed{\frac{3}{5}}$.

**83. The numbers 201, 204, 209, 216, 225, … are of the form $a_n = 200 + n^2$ where n = 1, 2, 3, 4, 5, …. For each n, let $D_n$ be the greatest common divisor of $a_n$ and $a_{n+1}$. What is the maximum value of $D_n$?**

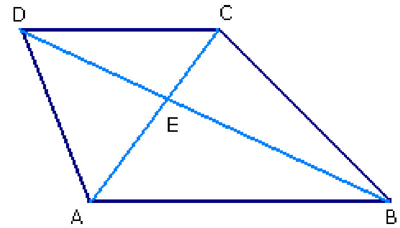
$HCF(a_n, a_{n+1}) = HCF(200 + n^2, 200 + (n + 1)^2) = HCF(200 + n^2, 200 + n^2 + 2n + 1) = HCF(200 + n^2, 2n + 1) = HCF(800 + 4n^2, 2n + 1) = HCF(800 - 2n, 2n + 1) = HCF(801, 2n + 1) = \boxed{801}$ when n = 400.

**84. Messers Baker, Cooper, Parson and Smith are a baker, a cooper, a parson and a smith. However, no one has the same name as his vocation. The cooper is not the namesake of Mr. Smith's vocation; the baker is neither Mr. Parson nor is he the namesake of Mr. Baker's vocation. What is Mr. Baker's vocation?**

There are two possible cases:

| | | |
|---|---|---|
| **Mr. Baker** | parson | parson |
| **Mr. Cooper** | baker | smith |
| **Mr. Parson** | smith | cooper |
| **Mr. Smith** | cooper | baker |

So in both the cases: Mr. Baker is **a parson**.




**85. In trapezium ABCD, AB||CD. If area(ΔABE) = $\log_a 11$, area(ΔCDE) = $\log_{11} a$, and area(ΔABC) = 11, find the area of ABCD.**

$[\text{area}(\Delta BEC)]^2 = [\text{area}(\Delta ABE)][\text{area}(\Delta CDE)] = 1$

=> area(ΔBEC) = 1 and area(ΔABE) = 10 and area(ΔCDE) = 1/10

=> area(ABCD) = $\left(\sqrt{10} + \frac{1}{\sqrt{10}}\right)^2 = \left(\frac{11}{\sqrt{10}}\right)^2 = \boxed{12.1}$.

**86. Let P(x) be a polynomial such that $P(x) = x^{19} - 2011x^{18} + 2011x^{17} - \dots - 2011x^2 + 2011x$. Calculate P(2010).**

$P(x) = (x^{19} - 2010x^{18}) - (x^{18} - 2010x^{17}) + (x^{17} - 2010x^{16}) - \dots - (x^2 - 2010x) + x$

So P(2010) = $\boxed{2010}$.

**87. How many 9-digit numbers (in decimal system) divisible by 11 are there in which every digit occurs except zero?**

Using divisibility rule of 11; difference between sums of alternate digits of the nine digit number must be divisible by 11. As there are 9 digits from 1 to 9, sum of all the digits is 45 and one group will be having 4 digits and other 5 digits. Also there difference will be 11 only. So sum of one group is 28 and other is 17. Now there are two cases: four digits a + b + c + d = 17 or 28.

For a + b + c + d = 17, we have (1, 2, 5, 9); (1, 3, 4, 9); (1, 2, 6, 8); (1, 3, 5, 8); (1, 3, 6, 7); (1, 4, 5, 7); (2, 3, 4, 8); (2, 3, 5, 7); (2, 4, 5, 6) i.e. 9 cases, and for a + b + c + d = 28, we have (4, 7, 8, 9); (5, 6, 8, 9) i.e. 2 cases. So total numbers are 11 • 4! • 5! = $\boxed{31680}$.

**88. There are four unit spheres inside a larger sphere, such that each of them touches the large sphere and the other three unit spheres. What is the radius of larger sphere?**

Analyze the structure as three unit spheres touching each other and the fourth one is now lying over them such that when centers of all the four unit spheres are joined they form a regular tetrahedron of edge length two units. Also observe that altitude from vertex of this tetrahedron on the opposite triangular face falls on its centroid.

So height of tetrahedron can be calculated as $\sqrt{2^2 - \left(\frac{2}{3}\sqrt{3}\right)^2} = 2\sqrt{\frac{2}{3}}$.

[*Remember length of altitude in an equilateral triangle of side 'a' is $\frac{\sqrt{3}}{2}a$ and centroid divides the altitude in the ratio 2 : 1 and also barycenter of a regular tetrahedron divides the height in the ratio 3 : 1*]

So radius of circumsphere of the tetrahedron is = $\frac{3}{4}\left(2\sqrt{\frac{2}{3}}\right) + 1 = \boxed{\sqrt{\frac{3}{2}} + 1}$.

**89. For the real numbers a, b and c, it is known that**

$$\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} = 1, \text{ and}$$

$$a + b + c = 1.$$

**Find the value of the expression, M = $\frac{1}{1+a+ab}+\frac{1}{1+b+bc}+\frac{1}{1+c+ca}$.**

By given two equation we find that a + b + c = abc = 1

Now M = $\frac{1}{1+a+ab}+\frac{1}{1+b+bc}+\frac{1}{1+c+ca}$ (1)

=> M = $\frac{1}{1+a+(1/c)}+\frac{1}{1+b+(1/a)}+\frac{1}{1+c+(1/b)}$

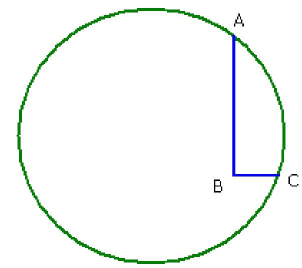
=> M = $\frac{c}{1+c+ac}+\frac{a}{1+a+ab}+\frac{b}{1+b+bc}$ (2)

=> M = $\frac{c}{1+c+(1/b)}+\frac{a}{1+a+(1/c)}+\frac{b}{1+b+(1/a)}$

=> M = $\frac{bc}{1+b+bc}+\frac{ca}{1+c+ca}+\frac{ab}{1+a+ab}$ (3)

Adding (1), (2) and (3), we get

3M = 3 => M = $\boxed{1}$.




**90. In the circle shown, radius = $\sqrt{50}$, AB = 6, BC = 2, ∠ABC = 90°. Find the distance from B to the centre of the circle.**

Extend AB to D and CB to E as shown.

Using AB × BD = BC × BE we have BE = 3 × BD

Let BD = 2x, then BE = 6x.

Let O be the centre of circle then drop perpendiculars from O to AD and CE to meet at their midpoints P and Q respectively as shown.

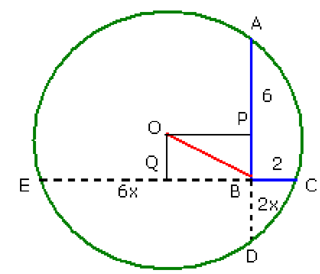
Using Pythagoras theorem in triangle OPA, we get

$OA^2 = OP^2 + AP^2$

=> $50 = (3x - 1)^2 + (3 + x)^2 = (9x^2 - 6x + 1) + (x^2 + 6x + 9) = 10x^2 + 10$

=> $10x^2 = 40$ and x = 2.

So OB = $\sqrt{OP^2+BP^2} = \sqrt{5^2+1^2} = \boxed{\sqrt{26}}$.




**91. Today is Friday. What day will it be after $4^{2010}$ days?**

$4^{2010} \equiv 4^{3\times670}$ mod7 ≡ $1^{670}$ mod7 ≡ 1 mod7. So it will be next to Friday i.e. **Saturday** after $4^{2010}$ days.

**92. Solve the congruence cryptarithm *LIFE* ≡ *SIZE* (mod *ELS*) in base 6 with E, L and S nonzero, all alphabets representing different numerals and Z > L > S.**
**Write the 6 letter-word denoting the digits 012345 as answer.**

As *LIFE* – *SIZE* ends in zero that means S must be 2 or 3 only and the quotient will be 3 or 2 respectively.

If S = 3, then L = 4 and Z = 5 are only values. It can be easily verified that it doesn't satisfy.

If S = 2, L = 3 that also doesn't satisfy. Now only case left is: S = 2, L = 4 and Z = 5 and it satisfies with the values as – 4103 ≡ 2153 mod342 i.e. 012345 ≡ ***FISELZ***.

**93. Find the sum $1+\frac{3}{2}+\frac{5}{4}+\frac{7}{8}+\ldots$**

Let S = $1+\frac{3}{2}+\frac{5}{4}+\frac{7}{8}+\ldots$

So S/2 = $\frac{1}{2}+\frac{3}{4}+\frac{5}{8}+\frac{7}{16}+\ldots$ Subtracting the two equations we get, S – S/2 = S/2 = 1 + 1 + ½ + ¼ + 1/8 + ...

=> $S/2 = 1 + \frac{1}{1 - 1/2} = 3$ and S = **6**.

**94. Two boats start at same instant from opposite ends of the river traveling across the water perpendicular to shores. Each travels at a constant but different speed. They pass at a point 720 meters from the nearest shore. Both boats remain at their slips for 15 minutes before starting back. On the return trip, they pass 400 meters from the other shore. Find the width of the river.**

When they meet for the first time, sum of the distances traveled = d (width of the river)
And when they meet in the return trip, sum of the distances traveled = 3d.
Because they are traveling with constant speeds and for equal time; total distance traveled by each of the boat till second meeting = 3 × distance traveled by that boat till first meeting
So d + 400 = 3 × 720 = 2160 => d = **1760 meters**.

**95. Larry, Curly, and Moe had an unusual combination of ages. The sum of any two of the three ages was the reverse of the third age (*e.g.*, 16 + 52 = 68, the reverse of 86). All were under 100 years old. What was the sum of the ages?**

Let the three numbers be ab, cd and ef where a, b, c, d, e, f are the single digit positive integers.
As cd + ef = ba, sum of the three numbers is: ab + ba = 11(a + b) = 11(c + d) = 11(e + f)
Also ab + cd = fe

- ⇨ 10a + b + 10c + d = 10f + e
- ⇨ 10(a + c - f) = e – b – d
- ⇨ a + c = f and e = b + d OR a + c = f – 1 and e + 10 = b + d

From equation above we have, (a + b) + (c + d) = 2(e + f)

- ⇨ a + c = 2(e + f) – (b + d)

So if a + c = f, then using e = b + d, we get f = e + 2f i.e. e + f = 0 which is not possible.
If a + c = f – 1, then using e + 10 = b + d, we get f – 1 = e + 2f – 10 i.e. e + f = 9.
(a) Thus a + b = c + d = e + f = 9 and sum of all the three ages = 11 × 9 = **99**.

**96. Find the sum of all four-digit numbers N whose sum of digits is equal to 2010 – N.**

N has to be greater than 2010 – 28 = 1982 because maximum sum of digits less than 2010 is 1 + 9 + 9 + 9 i.e. 28. Now checking for all the decades there are only two possible values of N i.e. 1986 and 2004. So the required sum is **3990**.

**97. DaGny has 11 different colors of fingernail polish. Find the number of ways she can paint the five fingernails on her left hand by using at least three colors such that no two consecutive finger nails have same color. Also she is to apply only one color at one fingernail which is quite unusual for her.**

First fingernail can be painted in 11 ways and every next fingernail can be painted in 10 ways each such that no two consecutive fingernails have same color i.e. total $11 \times 10^4$ ways. In this number of ways atleast two colors have been used. So we need to subtract the ways in which exactly two colors have been used which are 11 × 10.
Thus required numbers of ways are 110000 – 110 = **109890**.

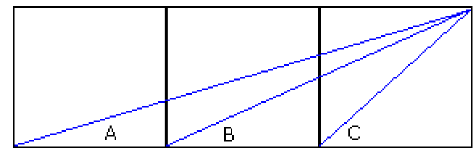
**98. A box contains 300 matches. Kamal and Sandeep take turns removing no more than half the matches in the box. The player who cannot move loses. What should be Kamal's first move to ensure his win if he is starting the game?**

Kamal must always leave N matches for Sandeep to continue where N is a natural number of the form $2^a - 1$. So Kamal must remove **45 matches** in his first move so as to leave 255 i.e. $2^8 - 1$ matches for Sandeep to continue with. Now whatever number of matches Sandeep removes Kamal will make Sandeep to continue with 127 matches. (Notice that Sandeep cannot remove more than 127 matches from 255.)

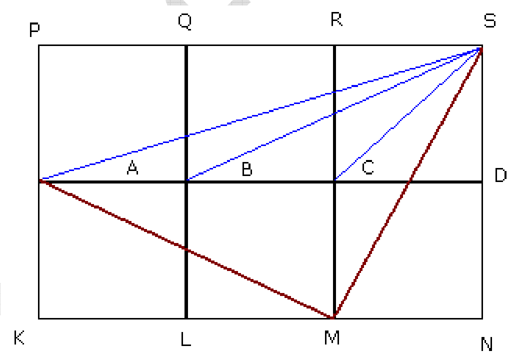
**99. Let N be an integer such that $2N^2$ has exactly 28 distinct positive divisors and $3N^2$ has exactly 24 distinct positive divisors. How many distinct positive divisors does $6N^2$ have?**

$N = 2 \times 3^3$, so $6N^2 = 2^3 \times 3^7$ and required number of divisors: $4 \times 8 = \boxed{32}$.

**100. Three unit squares are joined as shown. Find the measure of ∠A + ∠B + ∠C.**




Construct three more identical squares below the three given ones and draw two lines as shown on the right.




Now observe that triangle ASM is an isosceles right triangle, right angled at M. So ∠SAM = 45°.

Also triangle ACM ≅ BDS, so ∠B = ∠MAC.

=> ∠A + ∠B = 45°

As triangle CDS is also isosceles right triangle, ∠C = 45°.

Hence, ∠A + ∠B + ∠C = $\boxed{90°}$.

**101. What is the 625th term of the series where each term is made up of even digits only?**

**2, 4, 6, 8, 20, 22, 24, 26, 28, 40, 42, …**

625th term of the series is the smallest 5-digit number of the series i.e. $\boxed{20000}$.

**102. The houses in a street are spaced so that each house of one lane is directly opposite to a house of other lane. The houses are numbered 1, 2, 3, … and so on up one side, continuing the order back down the other side. Number 39 is opposite to 66. How many houses are there?**

Total number of houses = 39 + 66 – 1 = $\boxed{104}$.

**103. In a polygon, internal angles have the measures of 90° and 270° only. If there are 18 angles of measure 270°, then what is the number of angles with measure of 90°?**

Let there be n angles of measure 90°, then sum of all the angles = (90n) + (18 × 270) = (n + 16)180

So n = $\boxed{22}$.

**104. How many pair of positive integers (a, b) are there such that their LCM is 2012?**

$2012 = 2^2 \times 503$. Among a and b atleast one must have $2^2$ and 503. So the required number of ordered pairs (a, b) = $(3^2 - 2^2)(2^2 - 1^2) = \boxed{15}$.

**105. What is the sum of all natural numbers which are less than 2012 and co-prime to it?**

It is simply ½ × 2012 × ∅(2012) = $\boxed{1010024}$.

**106. How many positive integers N satisfy: (i) N < 1000 and (ii) $N^2 - N$ is divisible by 1000?**

$N \equiv 0$ or 1 $\bmod 2^3$ and also $N \equiv 0$ or 1 $\bmod 5^3$. So there are 3 positive integers i.e. 1, 376, and 625 which satisfy the given conditions.

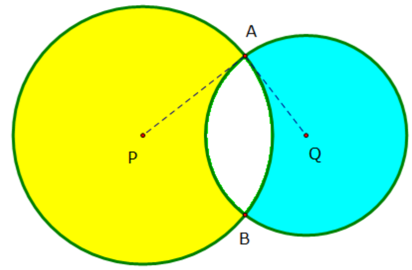
**107. 25 is a square number and can be written as average of two different square numbers i.e. 1 and 49. How many other square numbers from 1 to 625 inclusive can be written as average of two different square numbers?**

Only square of largest number of a Pythagorean triplet can be written in the desired form, as if $c^2 = a^2 + b^2$, then only we can write $2c^2 = (a + b)^2 + (a - b)^2$. So from $1^2$ to $25^2$, $c^2$ can take 6 values other than $5^2$ i.e. $10^2$, $13^2$, $15^2$, $17^2$, $20^2$ and $25^2$.

**108. I can break a block of 7 kg in smaller blocks of integral weights in four ways i.e. {1, 2, 4}, {1, 2, 2, 2}, {1, 1, 1, 4}, {1, 1, 1, 1, 1, 1, 1} such that I can measure each weight from 1**

**kg to 7 kg in exactly one way in either case. For example, using 1st case this is the only possible combination of weights to measure 1 kg to 7 kg: 1 = 1, 2 = 2, 3 = 1 + 2, 4 = 4, 5 = 1 + 4, 6 = 2 + 4, and 7 = 1 + 2 + 4. So find the number of ways a block of 14 kg can be broken under similar conditions e.g. {1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1} is a valid case but {1, 2, 3, 4, 4} is not.**
There are exactly 3 ways including the one mentioned in the problem statement. The other two are: {1, 1, 1, 1, 5, 5}, {1, 1, 3, 3, 3, 3}.

**109. I am twice as old as you were when I was as old as you are. What is the ratio of ages of mine and yours?**
4 : 3.




**110. Circles with centers P and Q have radii 20 and 15 cm respectively and intersect at two points A, B such that ÐPAQ = 90°. What is the difference in the area of two shaded regions?**
The required difference is = $\pi(20^2 - 15^2) = 175\pi$.

**111. What is the largest integer that is a divisor of**

$$(n + 1)(n + 3)(n + 5)(n + 7)(n + 9)$$

**for all positive even integers n?**
Given expression is product of five consecutive odd numbers, so it's always divisible by $1 \times 3 \times 5 = \boxed{15}$.

**112. For how many ordered pairs of positive integers (x, y), xy/(x+y) = 9?**
Simplifying we get, $(x - 9)(y - 9) = 81$. As $81 = 3^4$ has 5 positive integral divisors, number of ordered pairs of positive integers (x, y) is $\boxed{5}$.

**113. Amu, Bebe, Chanda and Dori played with a deck of 52 cards. In one game, Dori was dealing out the cards one by one to the players, starting with Amu, followed by Bebe, Chanda and Dori in this order, when suddenly some of the cards she had not dealt out yet slipped out of her hands and fell on the floor. The girls noticed that the number of cards on the floor was 2/3 of the number of cards Amu had already got, and the number of cards that Chanda had got was 2/3 of those in the remaining part of the deck in Dori's hand that she had not dealt out yet. How many cards had Dori dealt out altogether?**
Amu and Bebe had 9 cards each while Chanda and Dori had 8 cards each. 6 cards were fallen off and 12 are yet to be dealt. So altogether $9 + 9 + 8 + 8 = \boxed{34}$ cards had been dealt.

**114. In a city, 2/3 of the men and 3/5 of the women are married. (Everyone has one spouse and the spouses live in the same city.) What fraction of the inhabitants of the city is married?**
$\boxed{12/19}$.

**115. The sum of all interior angles of eight polygons is 3240°. What is the total number of sides of polygons?**
Let the number of sides in the polygons are: $n_1, n_2, \ldots, n_8$. So sum of all interior angles will be $(n_1 - 2)180° + (n_2 - 2)180° + \ldots + (n_8 - 2)180° = (n_1 + n_2 + \ldots + n_8)180° - 16*180° = 3240° = 18*180°$. So total number of sides of all polygons is simply $18 + 16 = \boxed{34}$. Remember that sum of all interior angles of an n-gon is $= (n - 2)180°$.

**116. Consider a triangle ABC with BC = 3. Choose a point D on BC such that BD = 2. Find the value of**

$$AB^2 + 2AC^2 - 3AD^2.$$

Using Stewart's theorem, we have $(1 \times AB^2) + (2 \times AC^2) = (1 + 2)(AD^2 + 1 \times 2)$. So $AB^2 + 2AC^2 - 3AD^2 = \boxed{6}$. Alternately we can use Pythagoras Theorem multiple times to get the result.

**117. Determine the number of divisors of $2012^8$ that are less than $2012^4$.**

$2012 = 2^2 \times 503$. So number of divisors of $2012^8 = 17 \times 9 = 153$ and less than $2012^4 = (153 - 1)/2 = \boxed{76}$ as $2012^4$ is square root of $2012^8$.

**118. How many numbers in the following sequence are prime numbers?**
**{1, 101, 10101, 1010101, 101010101, ……}**

When numbers of 1's in the number is even, then number is divisible by 101 for sure but for odd number of 1's, the number can be factorised as $\underbrace{10101...1}_{2n+1\ times\ 1's} = \underbrace{1111..111}_{2n+1\ times\ 1's} \times \underbrace{909090..9090}_{n-1\ times\ 90's}91$. So there is only $\boxed{1}$ prime number in the sequence i.e. 101.

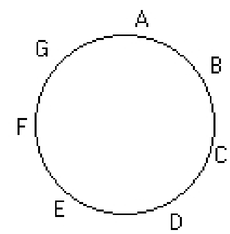
**119. How many triples of natural numbers (a, b, c) such that a, b and c are in geometric progression, and a + b + c = 111.**

There are total $\boxed{5}$ triplets and the solution triples are (37; 37; 37); (1; 10; 100); (100; 10; 1); (27; 36; 48); (48; 36; 27).

**120. What is the smallest integer n for which $\sqrt{n} - \sqrt{n-1} < 0.01$?**

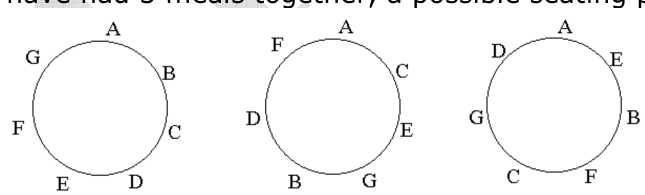
Given expression can be written as $\sqrt{n} + \sqrt{n-1} > 100$ i.e. $\sqrt{n} > 50 > \sqrt{n-1}$. So the smallest such n is = $\boxed{2501}$.

**121. Seven people, A, B, C, D, E, F and G can sit down for a meal at a round table as shown. Each person has two neighbours at the table: for example, A's neighbours are B and G. There are other ways in which the people can be seated round the table. Last month they dined together on a number of occasions, and no two of the people were neighbours more than once. How many meals could they have had together during the month?**



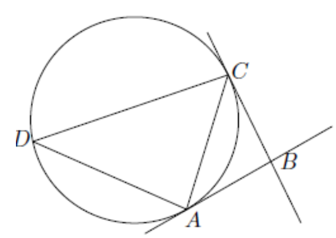

There are total 6!/2 = 3(5!) number of ways of seating arrangement as A-B is same as that of B-A. Also when A, B are together, they have been counted (5!) times. But as this pair has to be counted once only, we need to divide the total arrangements by (5!) to get the desired arrangements as = 3(5!)/(5!) = 3.

To show that it is possible to have had 3 meals together, a possible seating plan is:




Hence the greatest number of meals they could have had together during the month is $\boxed{3}$.

**122. Points A, D and C lie on the circumference of a circle. The tangents to the circle at points A and C meet at the point B. If ∠DAC = 83° and ∠DCA = 54°. Find ∠ABC.**




∠ADC = 180° - ∠DAC - ∠DCA = 180° - 83° - 54° = 43° = ∠ACB = ∠CAB (Alternate segment theorem).

So ∠ABC = 180° - 2(43°) = $\boxed{94°}$.

**123. How many 4-digit numbers uses exactly three different digits?**

Required 4-digit numbers will be of just all arrangements of a number of the form 'aabc' which has three distinct digits; a, b and c. So number of such numbers is given by $(9/10)(^{10}C_3)(^3C_1)(4!/2!) = \boxed{3888}$.

**124. The ratio of two six digit numbers abcabc and ababab is 55 : 54. Find the value of a + b + c.**

$$\frac{abcabc}{ababab} = \frac{1001 \times abc}{10101 \times ab} = \frac{7 \times 11 \times 13 \times abc}{3 \times 7 \times 13 \times 37 \times ab} = \frac{11 \times abc}{3 \times 37 \times ab} = \frac{55}{54}.$$

=> $\frac{abc}{ab} = \frac{5 \times 37}{18} = \frac{185}{18}$. So a + b + c = 1 + 8 + 5 = $\boxed{14}$.

**125. Find the infinite sum:**

$$\frac{1}{4} + \frac{1}{8} + \frac{2}{16} + \frac{3}{32} + \frac{5}{64} + \frac{8}{128} + \frac{13}{256} + \frac{21}{512} + \ldots$$

Let S = $\frac{1}{4} + \frac{1}{8} + \frac{2}{16} + \frac{3}{32} + \frac{5}{64} + \frac{8}{128} + \frac{13}{256} + \frac{21}{512} + \ldots$

So ½ S = $\frac{1}{8} + \frac{1}{16} + \frac{2}{32} + \frac{3}{64} + \frac{5}{128} + \frac{8}{256} + \frac{13}{512} + \frac{21}{1024} + \ldots$

Subtracting the two we get, ½ S = $\frac{1}{4} + \frac{1}{16} + \frac{1}{32} + \frac{2}{64} + \frac{3}{128} + \frac{5}{256} + \frac{8}{512} + \ldots = \frac{1}{4} + \frac{1}{4}S$.

Hence S = $\boxed{1}$.

**126. What is the probability of tossing a coin 6 times such that no two consecutive throws result in a head?**

If I write the favorable number of outcomes in a sequence for 1 toss, 2 tosses, 3 tosses, 4 tosses,….., there is a pattern. Just try to observe it. Let F(n) gives the favorable number of outcomes for n tosses and T(n) denotes total possible outcomes of n tosses. It is clear that T(n) = $2^n$. Now sequence of F(n) for n = 1, 2, 3, … is as follows: 2, 3, 5, …. It can be clearly inferred that F(n) is a Fibonacci sequence and F(6) = 21. So the required probability = F(6)/T(6) = $21/2^6$ = $\boxed{21/64}$.

**127. In how many ways 3 letters can be selected from 3 identical A's, 3 identical B's and 3 identical C's?**

It is simply the whole number solutions of A + B + C = 3 i.e. ${}^5C_2$ = $\boxed{10}$.

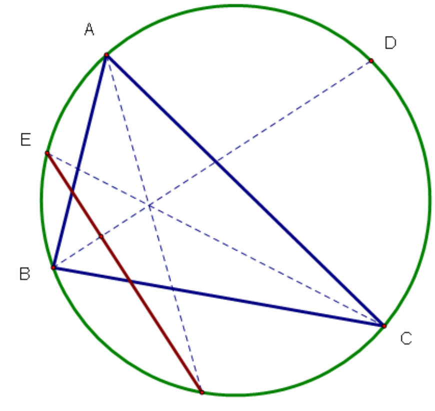
**128. For how many pairs of positive integers (x, y) both $x^2$ + 4y and $y^2$ + 4x are perfect squares?**

Let x ≤ y, then $y^2 < y^2 + 4x < y^2 + 4y < (y + 2)^2$.
So $y^2 + 4x = (y + 1)^2$ i.e. x = (2y + 1)/4 which is never possible. Hence $\boxed{0}$ pairs are there.

**129. Jar X contains six liters of a 46% milk solution; Jar Y contains three liters of a 43% milk solution and Jar Z contains one liter of p% milk solution. q/r liters of solution from Jar Z is transferred to Jar X and remaining solution from Jar Z is transferred to Jar Y such that resulting two solutions both contain 50% milk solution. Also q and r are positive integers co-prime to each other. Find the value of p + q + r.**

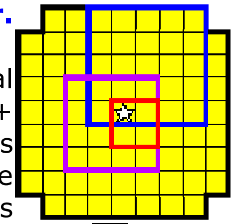
Total quantity of solution is 6 + 3 + 1 = 10 liters out of which 50% i.e. 5 liters is milk. So 0.46*6 + 0.43*3 + p/100 = 5. Simplifying we get *p = 95*.
Now let 'n' liters has been transferred from Jar Z to Jar X, so we get 0.46*6 + 0.95*n = 0.5(6 + n). Simplifying we get n = *q/r = 8/15*.
So p + q + r = 95 + 8 + 15 = $\boxed{118}$.

**130. ABC is an isosceles right triangle inscribed in a circle such that ∠B = 90°. BD, CE and AF are angle bisectors of triangle ABC as shown. What is the measure of smaller angle of intersection of BD and EF?**




Let I be the incenter and O be the intersection of BD and EF. In ▲IEO, ∠IEO = ∠CEF = ∠CAF = 22.5°. And ∠EIO = ∠ICB + ∠IBC = 22.5° + 45° = 67.5°. Hence ∠EOI = 180° - 22.5° - 67.5° = 90°. So EF ⊥ BD and required angle is $\boxed{90°}$.

**131. In the diagram, three squares are shown, all containing the star. Altogether, how many squares containing the star can be found in the diagram?**



If the given grid would have been a complete 9 × 9 grid, then because of symmetrical placement of star, required number of squares would have been $1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 4^2 + 3^2 + 2^2 + 1^2$ counting squares of unit length, two units, … up to nine units respectively. But as the grid given is not a complete square, we need to subtract the number of squares formed using the corner ones i.e. 4 + 4 + 4 + 4 + 1 counting squares of side-length five units, six units,.. nine units respectively. So the final answer comes out to be 68.

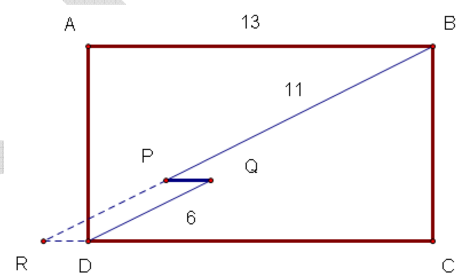
**132. Find all integers x, y, z (such that x < y < z) greater than 1 for which xy - 1 is divisible by z, yz - 1 is divisible by x, and zx - 1 is divisible by y.**

First point to note is that x, y, z are all co-prime. Also $(xy - 1)(yz - 1)(zx - 1) \equiv 0 \bmod(xyz)$

- ⇨ $(xyz)^2 - xyz(x + y + z) + xy + yz + zx - 1 \equiv 0 \bmod(xyz)$
- ⇨ $xy + yz + zx \equiv 1 \bmod(xyz)$
- ⇨ $xy + yz + zx > xyz$
- ⇨ $1/x + 1/y + 1/z > 1$

Only one triplet (x, y, z) ∈ (2, 3, 5) is possible.

**133. In a rectangle ABCD, AB = 13 and BC = 8. PQ lies inside the rectangle such that BP = 11, DQ = 6, AB||PQ, and BP||DQ. Find the length of PQ.**




First draw the diagram as shown (no other placement of PQ is possible). Extend BP to meet CD at R such that PQDR becomes a parallelogram with PQ = RD and DQ = PR = 6.

In right angle triangle BCR, CR = $\sqrt{BR^2 - BC^2} = \sqrt{17^2 - 8^2}$ = 15, so RD = PQ = CR – CD = 15 – 13 = 2.

**134. What are the last two digits of the sum obtained by adding all the possible remainders of numbers of the form $2^n$, n being a non-negative integer, when divided by 100?**

We are to find remainder of a GP of 2 with 100. Important point to note is that there will be a power of 2 after which the last two digits will start repeating. Let's say $2^a \equiv 2^b$ mod25, that means $2^{(a-b)} \equiv 1$ mod25 or there exists a *k* such that $2^k - 1$ is multiple of 25.

Now the number we are finding = N = $2^0 + 2^1 + 2^2(1 + 2 + \dots + 2^{k-1})$ mod100 ≡ (1 + 2) mod100 ≡ 03.

**135. On planet LOGIKA, there live two kinds of inhabitants; black and white ones and they answer every question posed to them in a Yes or No. Black inhabitants of northern hemisphere always lie while white inhabitants of northern hemisphere always tell the truth. Also white inhabitants of southern hemisphere always lie while black inhabitants of southern hemisphere always tell the truth. On a dark night, there is an electricity failure and you meet an inhabitant without knowing your location on the planet. What single yes/no question can you ask the inhabitant to determine color of the inhabitant?**

Are you a northerner? If he answers in Yes, he is White otherwise he is Black.

**136. N is product of first 50 prime numbers. A is a factor of N and B is a factor of A. How many ordered pairs (A, B) exist?**

Every prime number has three possibilities (1: contained in A & B, 2: contained in A and not in B, not contained in either of A, B). So number of ordered pairs of (A, B) = $3^{50}$.

**137. For how many positive integers, N > 2, (N - 2)! + (N + 2)! Is a perfect square?**

(N - 2)! + (N + 2)! = [(N + 2)(N + 1)(N)(N - 1) + 1](N - 2)!

We know that product of any four consecutive natural numbers is always 1 less than a perfect square. So the term in [] is always a perfect square and for the complete expression to be a perfect square – (N

- 2)! has to be a perfect square which is only possible for N = 3 as N has to be greater than 1. Hence there is only $\boxed{1}$ value of N which makes the expression a perfect square.

**138. Let P = $\frac{1}{1\times2}+\frac{1}{3\times4}+\frac{1}{5\times6}+\ldots+\frac{1}{2013\times2014}$**
**and Q = $\frac{1}{1008\times2014}+\frac{1}{1009\times2013}+\frac{1}{1010\times2012}+\ldots.+\frac{1}{2014\times1008}$. Find P/Q.**

$$P = \left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\ldots+\left(\frac{1}{2013}-\frac{1}{2014}\right)$$

$$P = \left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{2014}\right)-2\left(\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2014}\right) = \frac{1}{1008}+\frac{1}{1009}+\frac{1}{1010}+\ldots+\frac{1}{2014}.$$

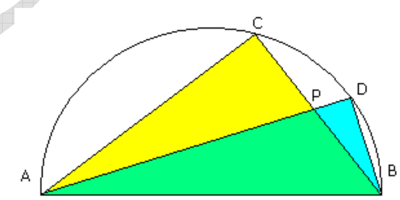
$$\text{So } 2P = \left(\frac{1}{1008}+\frac{1}{2014}\right)+\left(\frac{1}{1009}+\frac{1}{2013}\right)+\left(\frac{1}{1010}+\frac{1}{2012}\right)+\ldots+\left(\frac{1}{2014}+\frac{1}{1008}\right)$$

$$\text{Or } 2P = 3022\left(\frac{1}{1008\times2014}+\frac{1}{1009\times2013}+\frac{1}{1010\times2012}+\ldots+\frac{1}{2014\times1008}\right) = 3022Q.$$

Hence P/Q = $\boxed{1511}$.

**139. C and D are two points on a semicircle with AB as diameter such that AC – BC = 7 and AD – BD = 13. AD and BC intersect at P. Find the difference in area of triangles ACP and BDP.**

Difference in area of triangles ACP and BDP is same as difference in area of triangles ACB and BDA which is ½ *(AC*BC – AD*BD).




In ▲ABC, $(AC - BC)^2 = AC^2 + BC^2 - 2AC*BC = AB^2 - 2AC*BC = 7^2 = 49$.
And in ▲ABD, $(AD - BD)^2 = AD^2 + BD^2 - 2AD*BD = AB^2 - 2AD*BD = 13^2 = 169$.
Subtracting the above two equations we get, 2(AC*BC – AD*BD) = 169 – 49 = 120.
So ½ (AC*BC – AD*BD) = ¼ (120) = $\boxed{30}$.

**140. TG *fashions* hold its annual sale on the eve of Pi-day (14th March) and offered a discount of 90% on all its apparels. But this month it is offering the usual 80% discount. How much percent more I need to pay now than that on the annual sale's eve for purchase of similar clothing?**

Earlier I need to pay 10% of the price and now I'll have to pay 20% i.e. $\boxed{100\%}$ more price is to be paid.

**141. In how many ways 1,000,000 can be expressed as sum of a square number and a prime number?**

Let 1,000,000 = $1000^2 = a^2 + p$ where p is a prime number.
So $p = 1000^2 - a^2 = (1000 - a)(1000 + a)$.
As LHS is a prime number, (1000 – a) should be equal to 1 and s = 999. Now p comes out to be 1999 which is a prime number. So there is only $\boxed{1}$ desired way; 1,000,000 = $999^2 + 1999$.

**142. How many ordered triples of three positive integers (a, b, c) exist such that $a^3 + b^3 + c^3 = 2011$?**

For a positive integer N, $N^3 \equiv 0$ or $\pm1$ mod9. But $2011 \equiv 4$ mod9. So LHS can never be equal to RHS. Hence $\boxed{0}$ triples are there.

**143. In a quadrilateral ABCD, sides AD and BC are parallel but not equal and sides AB = DC = x. The area of the quadrilateral is 676 cm². A circle with centre O and radius 13 cm is inscribed in the quadrilateral such that it is tangent to each of the four sides of the quadrilateral. Determine the length of x.**

Given quadrilateral ABCD is an isosceles trapezium. Equating the lengths of common tangents in the trapezium, we get x = ½ (sum of parallel sides).
Also a key point to observe is that diameter of circle is acting as height of the trapezium which is 26cm.
We know that area of a trapezium = ½ (sum of parallel sides) (height of trapezium).
So putting all knowns and looking for unknowns, we get that 26x = 676 and x = 26.

**144. Kiran, Shashi and Rajni are Kiran's spouse, Shashi's sibling and Rajni's sister-in-law in no particular order. Also Kiran's spouse and Shashi's sibling are of same sex. Who among the three is a married male?**
Rajni's sister-in-law is certainly female. And other two are of same sex. Certainly they cannot be females (because there is a married couple also). So Kiran's spouse and Shashi's sibling are male. That means Kiran is a female while Rajni and Shashi are males. And Shashi is the married male i.e. husband of Kiran.

**145. A and B start running from two opposite ends of a 1000m racing track. A and B travel with a speed of 8m/s and 5m/s respectively. How many times they meet, while running, in first 1000s after start?**
In first 1000s after start, A takes 8 end-to-end journey while B takes 5. In every one of his end-to-end journey, A meets B exactly once. So, in all, they meet 8 times including the final meeting after 1000s.

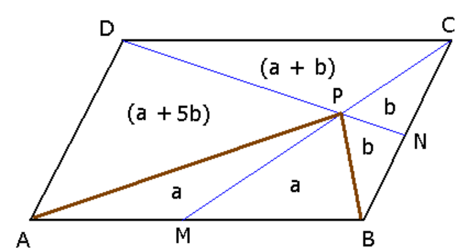
**146. According to death-will of Mr. Ranjan, all of this money was to be divided among his children in the following manner: `N to the first born plus 1/17 of what remains, `2N to the second born plus 1/17 of what then remains, `3N to the third born plus 1/17 of what then remains, and so on. When the distribution of the money was complete, each child received the same amount and no money was left over. Determine the number of children.**
Let there be total x children, then equating the amount received by $(x-1)^{th}$ and $x^{th}$ child we get, (x - 1)N + xN/16 = xN. Simplifying we get x = 16.

**147. One number is removed from the set of integers from 1 to n. The average of the remaining numbers is 40.75. Which integer was removed?**
Total numbers should a number of the form 4k + 1 near 80. So most probable such number is 81. Sum of first 81 natural numbers is 3321. Sum of given 80 numbers is 40.75*80 = 3260, so number 61 was removed.

**148. What is the $2037^{th}$ positive integer that can be expressed as the sum of two or more consecutive positive integers? (The first three are 3 = 1+2, 5 = 2+3, and 6 = 1+2+3.)**
All positive integers except powers of 2 can be written as sum of two or more consecutive positive integers. So the $2037^{th}$ such number is 2049.

**149. Determine the number of ordered triplets (A, B, C) of sets which have the property that**
**(i) $A \cup B \cup C$ = {1, 2, 3… 1000}, and**
**(ii) $A \cap B \cap C = \varnothing$.**
Every element of the set {1, 2, 3 … 1000} has exactly 6 choices viz. to be included in A, B, C, A&B, B&C, or A&C. Hence the required number of triplets = $6^{1000}$.

**150. In a parallelogram ABCD, let M be the midpoint of the side AB and N the midpoint of BC. Let P be the intersection point of the lines MC and ND. Find the ratio of area of ▲s APB : BPC : CPD : DPA.**
As M and N are mid points of AB and BC respectively, areas of ▲s APM, BPM and BPN, CPN are equal and equal to a, a and b, b respectively (say) as shown in the diagram. Also area(BMC) = area(CND) = ¼ area(ABCD) = a + 2b. So area(CPD) = a + b and area(DPA) = 4(a +



2b) – (a + a + b + b + a + b) = a + 5b. Now we have that area(APB) + area(CPD) = area(BPC) + area(DPA) = ½ area(ABCD) so we get that a = 3b. Hence ratio of area of ▲s APB : BPC : CPD : DPA = 3 : 1 : 2 : 4.

**151. Kali-Jot is a game played by two players each of them having some number of marbles with her. One of the two players has to determine whether the number of marbles with other player is even or odd. A particular game of Kali-Jot has seven players and starts with players $P_1$ and $P_2$ on field and the other players $P_3$, $P_4$, $P_5$, $P_6$, $P_7$ waiting in a queue for their turn in order. After each game is played, the loser goes to the end of the queue; the winner adds 1 point to her score and stays on the field; and the player at the head of the queue comes on to contest the next point. Game continues until someone has scored 11 points. At that moment, it was found out that a total of 43 points have been scored by all seven players together. Who is the winner?**

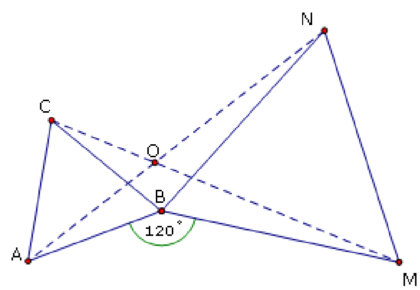
Whenever a player loses a game, she has to wait for five games before her next turn. If a is the number of games before her first turn, then the player will win if a + 6b + 11 = 43, where b ≥ 0 is an integer and 0 ≤ a ≤ 5. Here b is the number of times she lost. See whenever she lost, from that point onward till her next turn – 6 games has been played or 6 points have been distributed. From this, we obtain a = 2 and b = 5. Thus the second player in the queue wins. That is $P_4$ wins. 120°

**152. For positive real numbers; A, B, C, D such that A + B + C + D = 8, find the minimum possible value of $\frac{1}{A}+\frac{1}{B}+\frac{1}{C}+\frac{1}{D}$.**

For n positive real numbers ($a_1$, $a_2$, …$a_n$) we have $(a_1+a_2+\ldots+a_n)\left(\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_n}\right) \geq n^2$. So here we have that $\frac{1}{A}+\frac{1}{B}+\frac{1}{C}+\frac{1}{D} \geq 4^2/8 = 2$. Hence required minimum value is 2.

**153. In two equilateral triangles ABC and BMN, ∠ABM = 120°. AN & CM intersects at O. Find ∠MON.**

▲ABN ~ ▲CBM (S-A-S similarity), that means ∠ANB = ∠CMB. So O, B, M, N are con-cyclic and ∠MON = ∠MBN = 60°.



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