4). $\lim_{x\to 0}\frac{\sqrt{x^2+1}-\cos 2x}{x^2}$. Đặt $f(x)=\frac{\sqrt{x^2+1}-\cos 2x}{x^2}$

$$\lim_{x\to 0}\frac{\sqrt{x^2+1}-1+1-\cos 2x}{x^2}=\lim_{x\to 0}\frac{\sqrt{x^2+1}-1}{x^2}+\lim_{x\to 0}\frac{1-\cos 2x}{x^2}$$

- Tính $\lim_{x\to 0}\frac{\sqrt{x^2+1}-1}{x^2}=\lim_{x\to 0}\frac{x^2+1-1}{x^2\left(\sqrt{x^2+1}+1\right)}=\lim_{x\to 0}\frac{1}{\sqrt{x^2+1}+1}=\frac{1}{2}$

- Tính $\lim_{x\to 0}\frac{1-\cos 2x}{x^2}=\lim_{x\to 0}\frac{2\sin^2 x}{x^2}=2\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2=2$

Vậy $\lim_{x\to 0}f(x)=\frac{1}{2}+2=\frac{5}{2}$

5). $L=\lim_{x\to 0}\frac{\sqrt{1+2x}-\cos x-x}{x^2}$. Đặt $f(x)=\frac{\sqrt{1+2x}-\cos x-x}{x^2}$

$$L=\lim_{x\to 0}\frac{\sqrt{1+2x}-(1+x)+1-\cos x}{x^2}=\lim_{x\to 0}\frac{\sqrt{1+2x}-(1+x)}{x^2}+\lim_{x\to 0}\frac{1-\cos x}{x^2}$$

- Tính $\lim_{x\to 0}\frac{\sqrt{1+2x}-(1+x)}{x^2}=\lim_{x\to 0}\frac{1+2x-(1+x)^2}{x^2\left(\sqrt{1+2x}-(1+x)\right)}$

$$=\lim_{x\to 0}\frac{-x^2}{x^2\left(\sqrt{1+2x}-(1+x)\right)}=\lim_{x\to 0}\frac{-1}{\left(\sqrt{1+2x}+(1+x)\right)}=-\frac{1}{2}$$

- Tính $\lim_{x\to 0}\frac{1-\cos x}{x^2}=\lim_{x\to 0}\frac{2\sin^2\frac{x}{2}}{x^2}=\lim_{x\to 0}\frac{1}{2}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2=\frac{1}{2}$

Vậy $\lim_{x\to 0}f(x)=-\frac{1}{2}+\frac{1}{2}=0$.

6). $L=\lim_{x\to 0}\frac{\sqrt[3]{2x+1}-\sqrt{1-x}}{\sin 2x}$. Đặt $f(x)=\frac{\sqrt[3]{2x+1}-\sqrt{1-x}}{\sin 2x}$

$$L=\lim_{x\to 0}\frac{\sqrt[3]{2x+1}-1+1-\sqrt{1-x}}{\sin 2x}=\lim_{x\to 0}\frac{\sqrt[3]{2x+1}-1}{\sin 2x}+\lim_{x\to 0}\frac{1-\sqrt{1-x}}{\sin 2x}$$

- Tính $\lim_{x\to 0}\frac{\sqrt[3]{2x+1}-1}{\sin 2x}=\lim_{x\to 0}\frac{2x+1-1}{\sin 2x\left[\left(\sqrt[3]{2x+1}\right)^2+\sqrt[3]{2x+1}+1\right]}$

$=\lim_{x\to 0}\frac{2x}{2\sin x\cos x\left[\left(\sqrt[3]{2x+1}\right)^2+\sqrt[3]{2x+1}+1\right]}=\lim_{x\to 0}\frac{x}{\sin x}\cdot\frac{1}{\cos x\left[\left(\sqrt[3]{2x+1}\right)^2+\sqrt[3]{2x+1}+1\right]}=\frac{1}{3}$

- Tính $\lim_{x\to 0}\frac{1-\sqrt{1-x}}{\sin 2x}=\lim_{x\to 0}\frac{x}{2\sin x\cos x\left(1+\sqrt{1-x}\right)}=\lim_{x\to 0}\frac{x}{\sin x}\cdot\frac{1}{2\cos x\left(1+\sqrt{1-x}\right)}=\frac{1}{4}$

Vậy $\lim_{x\to 0}f(x)=\frac{1}{3}+\frac{1}{4}=\frac{7}{12}$

7). $\lim_{x\to\frac{\pi}{3}}\frac{\sin x-\sqrt{3}\cos x}{\sin 3x}=\lim_{x\to\frac{\pi}{3}}\frac{\sin x-\sqrt{3}\cos x}{3\sin x-4\sin^3 x}=\frac{\sin^2 x-3\cos^2 x}{\sin x\left(3-4\sin^2 x\right)\left(\sin x+\sqrt{3}\cos x\right)}$

$=\frac{4\sin^2 x-3}{\sin x\left(3-4\sin^2 x\right)\left(\sin x+\sqrt{3}\cos x\right)}=\frac{-1}{\sin x\left(\sin x+\sqrt{3}\cos x\right)}=\frac{-2}{3}$

8). $L=\lim_{x\to 0}\frac{1-\cos x\sqrt{\cos 2x}}{x^2}$

$L=\lim_{x\to 0}\frac{1-\cos x\sqrt{\cos 2x}}{x^2}=\lim_{x\to 0}\frac{1-\cos^2 x\cos 2x}{x^2\left(1+\cos x\sqrt{\cos 2x}\right)}$

$=\lim_{x\to 0}\frac{\cos^2 x+\sin^2 x-\cos^2 x\cos 2x}{x^2\left(1+\cos x\sqrt{\cos 2x}\right)}=\lim_{x\to 0}\frac{\cos^2 x\left(1-\cos 2x\right)+\sin^2 x}{x^2\left(1+\cos x\sqrt{\cos 2x}\right)}$

$=\lim_{x\to 0}\frac{2\sin^2 x\cos^2 x+\sin^2 x}{x^2\left(1+\cos x\sqrt{\cos 2x}\right)}=\lim_{x\to 0}\frac{\sin^2 x}{x^2}\cdot\frac{2\cos^2 x+1}{1+\cos x\sqrt{\cos 2x}}=\frac{3}{2}$

9). $L=\lim_{x\to 0}\frac{1-\sqrt[3]{\cos x}}{\tan^2 x}$

$L=\lim_{x\to 0}\frac{1-\cos x}{\tan^2 x\left[1+\sqrt[3]{\cos x}+\left(\sqrt[3]{\cos x}\right)^2\right]}=\lim_{x\to 0}\frac{2\sin^2\frac{x}{2}\cos^2 x}{4\sin^2\frac{x}{2}\cos^2\frac{x}{2}\left[1+\sqrt[3]{\cos x}+\left(\sqrt[3]{\cos x}\right)^2\right]}$

$$=\lim_{x\to 0}\frac{\cos^2 x}{2\cos^2\frac{x}{2}\left[1+\sqrt[3]{\cos x}+\left(\sqrt[3]{\cos x}\right)^2\right]}=\frac{1}{6}.$$

Câu 5: Tìm các giới hạn sau:

1). $\lim_{x\to\frac{\pi}{4}}\frac{\sqrt[3]{\tan x}-1}{2\sin^2 x-1}$ 2). $\lim_{x\to 0}\frac{1-\cos x\cos 2x}{x^2}$ 3). $\lim_{x\to 0}\left(\frac{2}{\sin 2x}-\cot x\right)$

4). $\lim_{x\to 0}\frac{1-\sqrt{2x+1}+\sin x}{\sqrt{3x+4}-2-x}$ 5). $\lim_{x\to 0}\frac{\sqrt{x^2+1}-\cos x}{x^2}$ 6). $L=\lim_{x\to 0}\frac{1-\sin 2x-\cos 2x}{1+\sin 2x-\cos 2x}$

7). $\lim_{x\to\frac{\pi}{3}}\frac{\cos 3x+2\cos 2x+2}{\sin 3x}$ 8). $\lim_{x\to 0}\frac{\cos\left(\frac{\pi}{2}\cos x\right)}{\sin^2\frac{x}{2}}$ 9). $\lim_{x\to 0}\frac{1-\cos x}{\left(1-\sqrt{1-x}\right)^2}$

LỜI GIẢI

1). $L=\lim_{x\to\frac{\pi}{4}}\frac{\sqrt[3]{\tan x}-1}{2\sin^2 x-1}$

$$L=\lim_{x\to\frac{\pi}{4}}\frac{\tan x-1}{\left(\sin^2 x-\cos^2 x\right)\left[\left(\sqrt[3]{\tan x}\right)^2+\sqrt[3]{\tan x}+1\right]}$$

$$=\lim_{x\to\frac{\pi}{4}}\frac{\sin x-\cos x}{\cos x\left(\sin x-\cos x\right)\left(\sin x+\cos x\right)\left[\left(\sqrt[3]{\tan x}\right)^2+\sqrt[3]{\tan x}+1\right]}$$

$$=\lim_{x\to\frac{\pi}{4}}\frac{1}{\cos x\left(\sin x+\cos x\right)\left[\left(\sqrt[3]{\tan x}\right)^2+\sqrt[3]{\tan x}+1\right]}=\frac{1}{3}$$

2). $\lim_{x\to 0}\frac{1-\cos x\cos 2x}{x^2}$

$$=\lim_{x\to 0}\frac{\sin^2 x+\cos^2 x-\cos x\cos 2x}{x^2}=\lim_{x\to 0}\frac{\sin^2 x}{x^2}+\lim_{x\to 0}\frac{\cos x\left(\cos x-\cos 2x\right)}{x^2}$$

$$=1+\lim_{x\to 0}\frac{2\cos x\sin\frac{3x}{2}\sin\frac{x}{2}}{x^2}=1+\lim_{x\to 0}\left(\frac{\sin\frac{3x}{2}}{\frac{3x}{2}}\right)\cdot\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)\cdot\frac{9}{8}\cdot\cos x=1+\frac{9}{8}=\frac{17}{8}$$

3). $L=\lim_{x\to 0}\left(\frac{2}{\sin 2x}-\cot x\right)$

$$L=\lim_{x\to 0}\left(\frac{1}{\sin x\cos x}-\frac{\cos x}{\sin x}\right)=\lim_{x\to 0}\frac{1-\cos^2 x}{\sin x\cos x}=\lim_{x\to 0}\frac{\sin^2 x}{\sin x\cos x}=\lim_{x\to 0}\tan x=0$$

4). $L=\lim_{x\to 0}\frac{1-\sqrt{2x+1}+\sin x}{\sqrt{3x+4}-2-x}$

$$L=\lim_{x\to 0}\frac{1-\sqrt{2x+1}+\sin x}{\sqrt{3x+4}-2-x}=\lim_{x\to 0}\frac{1-\sqrt{2x+1}}{\sqrt{3x+4}-2-x}+\lim_{x\to 0}\frac{\sin x}{\sqrt{3x+4}-2-x}$$

$$=\lim_{x\to 0}\frac{-2x\left(\sqrt{3x+4}+2+x\right)}{\left(-x^2-x\right)\left(1+\sqrt{2x+1}\right)}+\lim_{x\to 0}\frac{\left(\sqrt{3x+4}+2+x\right)\sin x}{-x^2-x}$$

$$=\lim_{x\to 0}\frac{2\left(\sqrt{3x+4}+2+x\right)}{\left(x+1\right)\left(1+\sqrt{2x+1}\right)}+\lim_{x\to 0}\frac{\sin x}{x}\cdot\frac{\sqrt{3x+4}+2+x}{-x-1}=4-4=0$$

5). $\lim_{x\to 0}\frac{\sqrt{x^2+1}-\cos x}{x^2}$

$$I=\lim_{x\to 0}\frac{\left(\sqrt{1+x^2}-1\right)+(1-\cos x)}{x^2}=\lim_{x\to 0}\left[\frac{\left(\sqrt{1+x^2}-1\right)}{x^2}+\frac{1-\cos x}{x^2}\right]$$

$$=\lim_{x\to 0}\left(\frac{x^2}{x^2(\sqrt{1+x^2}+1)}+\frac{2\sin^2\frac{x}{2}}{x^2}\right)=\lim_{x\to 0}\left(\frac{1}{\sqrt{1+x^2}+1}+\frac{\sin^2\frac{x}{2}}{2\frac{x^2}{4}}\right)=1$$

6). $L=\lim_{x\to 0}\frac{1-\sin 2x-\cos 2x}{1+\sin 2x-\cos 2x}$

$$L=\lim_{x\to 0}\frac{1-\sin 2x-\cos 2x}{1+\sin 2x-\cos 2x}=\lim_{x\to 0}\frac{1-\cos 2x-\sin 2x}{1-\cos 2x+\sin 2x}=\lim_{x\to 0}\frac{2\sin^2 x-2\sin x\cos x}{2\sin^2 x+2\sin x\cos x}$$

$$=\lim_{x\to 0}\frac{2\sin x\left(\sin x-\cos x\right)}{2\sin x\left(\sin x+\cos x\right)}=\lim_{x\to 0}\frac{\sin x-\cos x}{\sin x+\cos x}=-1$$

7). $\lim_{x\to\frac{\pi}{3}}\frac{\cos 3x+2\cos 2x+2}{\sin 3x}=\lim_{x\to\frac{\pi}{3}}\frac{4\cos^3 x+4\cos^2 x-3\cos x}{3\sin x-4\sin^3 x}$

$=\lim_{x\to\frac{\pi}{3}}\frac{\cos x\left(2\cos x+3\right)\left(2\cos x-1\right)}{\sin x\left(2\cos x-1\right)\left(2\cos x+1\right)}=$

8). $\lim_{x\to 0}\frac{\cos\left(\frac{\pi}{2}\cos x\right)}{\sin^2\frac{x}{2}}$

$=\lim_{x\to 0}\frac{\sin\left(\frac{\pi}{2}-\frac{\pi}{2}\cos x\right)}{\sin^2\frac{x}{2}}=\lim_{x\to 0}\frac{\sin\left(\frac{\pi}{2}2\sin^2\frac{x}{2}\right)}{\sin^2\frac{x}{2}}=\lim_{x\to 0}\left[\pi.\frac{\sin\left(\pi\sin^2\frac{x}{2}\right)}{\pi\sin^2\frac{x}{2}}\right]=\pi$

9). $\lim_{x\to 0}\frac{1-\cos x}{\left(1-\sqrt{1-x}\right)^2}$

$=\lim_{x\to\infty}\frac{\left(1-\cos x\right)\left(1+\sqrt{1-x}\right)^2}{\left(1-\sqrt{1-x}\right)^2\left(1+\sqrt{1-x}\right)^2}=\lim_{x\to 0}\frac{2\sin^2\left(\frac{x}{2}\right)\left(1+\sqrt{1-x}\right)^2}{4.\left(\frac{x}{2}\right)^2}=\lim_{x\to 0}\frac{\left(1+\sqrt{1-x}\right)^2}{2}=2$