$$=\lim_{x\to0}\left(\frac{\sin x}{x}\right)^2\left(\frac{2\sin(a+x)}{\cos(a+2x)\cos(a+x)\cos a}\right)=\frac{2\sin a}{\cos^3 a}.$$

Câu 3: Tìm các giới hạn sau:

1). $\lim\limits_{x\to0}\frac{\sin ax+\tan bx}{(a+b)x}\quad (a+b\neq0)$

2). $\lim\limits_{x\to0}\frac{\cos3x-\cos5x.\cos7x}{x^2}$

3). $\lim\limits_{x\to0}\frac{\cos ax-\cos bx.\cos cx}{x^2}$

4). $\lim\limits_{x\to0}\frac{\sin(a+x)-\sin(a-x)}{\tan(a+x)-\tan(a-x)}$

5). $\lim\limits_{x\to0}\frac{\sqrt{2x+1}-\sqrt[3]{x^2+1}}{\sin x}$

6). $\lim\limits_{x\to0}\frac{\sin^2 2x-\sin x.\sin4x}{x^4}$

7). $\lim\limits_{x\to0}\frac{1-\cos5x.\cos7x}{\sin^2 11x}$

8). $\lim\limits_{x\to0}\left(\frac{1}{\sin x}-\frac{1}{\tan x}\right)$

9). $\lim\limits_{x\to0}\frac{\sin x-\sin2x}{x\left(1-2\sin^2\frac{x}{2}\right)}$

10). $\lim\limits_{x\to0}\frac{\sqrt{1+x^2}-\cos x}{x^2}$

LỜI GIẢI

1). $$\lim_{x\to0}\frac{\sin ax+\tan bx}{(a+b)x}=\lim_{x\to0}\frac{\sin ax+\frac{\sin bx}{\cos bx}}{(a+b)x}=\lim_{x\to0}\frac{\sin ax}{(a+b)x}+\lim_{x\to0}\frac{\sin bx}{(a+b)x.\cos bx}$$

$$=\lim_{x\to0}\frac{a}{a+b}\cdot\frac{\sin ax}{ax}+\lim_{x\to0}\frac{b}{(a+b)\cos bx}\cdot\frac{\sin bx}{bx}=\frac{a}{a+b}+\frac{b}{a+b}=1$$

2). $$\lim_{x\to0}\frac{\cos3x-\cos5x.\cos7x}{x^2}=\lim_{x\to0}\frac{\cos3x-1+(1-\cos5x)\cos7x+1-\cos7x}{x^2}$$

$$=\lim_{x\to0}\frac{\cos3x-1}{x^2}\lim_{x\to0}\frac{(1-\cos5x)\cos7x}{x^2}+\lim_{x\to0}\frac{1-\cos7x}{x^2}$$

$$=\lim_{x\to0}\frac{-2\sin^2\frac{3x}{2}}{x^2}+\lim_{x\to0}\frac{2\sin^2\frac{5x}{2}\cos7x}{x^2}+\lim_{x\to0}\frac{2\sin^2\frac{7x}{2}}{x^2}$$

$$=\lim_{x\to0}\frac{-9}{2}.\left(\frac{\sin\frac{3x}{2}}{\frac{3x}{2}}\right)^2+\lim_{x\to0}\frac{25\cos7x}{2}\left(\frac{\sin\frac{5x}{2}}{\frac{5x}{2}}\right)^2+\lim_{x\to0}\frac{49}{2}\left(\frac{\sin\frac{7x}{2}}{\frac{7x}{2}}\right)^2=-\frac{9}{2}+\frac{25}{2}+\frac{49}{2}=\frac{65}{2}$$

3). $\lim\limits_{x\to0}\dfrac{\cos ax-\cos bx.\cos cx}{x^2}=\lim\limits_{x\to0}\dfrac{\cos ax-1-(\cos bx-1)\cos cx+1-\cos cx}{x^2}$

$=\lim\limits_{x\to0}\dfrac{-2\sin^2\frac{ax}{2}}{x^2}+\lim\limits_{x\to0}\dfrac{2\sin^2\frac{bx}{2}\cos cx}{x^2}+\lim\limits_{x\to0}\dfrac{2\sin^2\frac{cx}{2}}{x^2}$

$=\lim\limits_{x\to0}\dfrac{-a^2}{2}\left(\dfrac{\sin\frac{ax}{2}}{\frac{ax}{2}}\right)^2+\lim\limits_{x\to0}\dfrac{b^2\cos cx}{2}\cdot\left(\dfrac{\sin\frac{bx}{2}}{\frac{bx}{2}}\right)^2+\lim\limits_{x\to0}\dfrac{c^2}{2}\cdot\left(\dfrac{\sin\frac{cx}{2}}{\frac{cx}{2}}\right)^2=\dfrac{-a^2+b^2+c^2}{2}$

4). $\lim\limits_{x\to0}\dfrac{\sin(a+x)-\sin(a-x)}{\tan(a+x)-\tan(a-x)}=\lim\limits_{x\to0}\dfrac{2\cos a\sin x}{\dfrac{\sin 2x}{\cos(a+x)\cos(a-x)}}$

$=\lim\limits_{x\to0}\dfrac{\cos a\cos(a+x)\cos(a-x)}{\cos x}=\cos^3 a$

5). $\lim\limits_{x\to0}\dfrac{\sqrt{2x+1}-\sqrt[3]{x^2+1}}{\sin x}$

$\lim\limits_{x\to0}\dfrac{\sqrt{2x+1}-1+1-\sqrt[3]{x^2+1}}{\sin x}=\lim\limits_{x\to0}\dfrac{\sqrt{2x+1}-1}{\sin x}+\lim\limits_{x\to0}\dfrac{1-\sqrt[3]{x^2+1}}{\sin x}$

$=\lim\limits_{x\to0}\dfrac{2x}{\sin x\left(\sqrt{2x+1}+1\right)}+\lim\limits_{x\to0}\dfrac{-x^2}{\sin x\left[1+\sqrt[3]{x^2+1}+\left(\sqrt[3]{x^2+1}\right)^2\right]}$

$=\lim\limits_{x\to0}\dfrac{x}{\sin x}\cdot\dfrac{2}{\sqrt{2x+1}+1}+\lim\limits_{x\to0}\dfrac{x}{\sin x}\cdot\dfrac{-x}{1+\sqrt[3]{x^2+1}+\left(\sqrt[3]{x^2+1}\right)^2}=\dfrac{2}{1+1}+0=1$

6). $\lim\limits_{x\to0}\dfrac{\sin^2 2x-\sin x.\sin 4x}{x^4}=\lim\limits_{x\to0}\dfrac{\sin^2 2x-2\sin x\sin 2x\cos 2x}{x^4}$

$=\lim\limits_{x\to0}\dfrac{\sin 2x\left(2\sin x\cos x-2\sin x\cos 2x\right)}{x^4}$

$=\lim\limits_{x\to0}\dfrac{2\sin 2x.\sin x\left(\cos x-\cos 2x\right)}{x^4}=\lim\limits_{x\to0}\dfrac{4\sin 2x.\sin x.\sin\frac{3x}{2}.\sin\frac{x}{2}}{x^4}$

$$= \lim_{x\to 0} 6\cdot\left(\frac{\sin 2x}{2x}\right)\cdot\left(\frac{\sin x}{x}\right)\cdot\left(\frac{\sin\frac{3x}{2}}{\frac{3x}{2}}\right)\cdot\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right) = 6$$

7). $\lim_{x\to 0}\frac{1-\cos 5x.\cos 7x}{\sin^2 11x}$

$$= \lim_{x\to 0}\frac{(1-\cos 5x)\cos 7x + 1-\cos 7x}{\sin^2 11x} = \lim_{x\to 0}\frac{2\sin^2\frac{5x}{2}\cos 7x}{\sin^2 11x} + \lim_{x\to 0}\frac{2\sin^2\frac{7x}{2}}{\sin^2 11x}$$

$$= \lim_{x\to 0}\frac{\left(\frac{\sin\frac{5x}{2}}{\frac{5x}{2}}\right)^2\cos 7x}{\left(\frac{\sin 11x}{11x}\right)^2}\cdot\frac{25}{484} + \lim_{x\to 0}\frac{\left(\frac{\sin\frac{7x}{2}}{\frac{7x}{2}}\right)^2}{\left(\frac{\sin 11x}{11x}\right)^2}\cdot\frac{49}{484} = \frac{25}{484}+\frac{49}{484} = \frac{37}{242}$$

8). $\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{\tan x}\right) = \lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{\cos x}{\sin x}\right) = \lim_{x\to 0}\frac{1-\cos x}{\sin x} = \lim_{x\to 0}\frac{2\sin^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}$

$$= \lim_{x\to 0}\tan\frac{x}{2} = 0.$$

9). $\lim_{x\to 0}\frac{\sin x-\sin 2x}{x\left(1-2\sin^2\frac{x}{2}\right)} = \lim_{x\to 0}\frac{2\cos\frac{3x}{2}\sin\frac{-x}{2}}{x\cos x} = \lim_{x\to 0}\frac{\sin\frac{x}{2}}{\frac{x}{2}}\cdot\frac{-\cos\frac{3x}{2}}{\cos x} = -1$

10). $\lim_{x\to 0}\frac{\sqrt{1+x^2}-\cos x}{x^2} = \lim_{x\to 0}\frac{\sqrt{1+x^2}-1+1-\cos x}{x^2} = \lim_{x\to 0}\frac{\sqrt{1+x^2}-1}{x^2} + \lim_{x\to 0}\frac{1-\cos x}{x^2}$

$$= \lim_{x\to 0}\frac{x^2}{x^2\left(\sqrt{1+x^2}+1\right)} + \lim_{x\to 0}\frac{2\sin^2\frac{x}{2}}{x^2} = \lim_{x\to 0}\frac{1}{\sqrt{1+x^2}+1} + \lim_{x\to 0}\frac{1}{2}\cdot\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2 = \frac{1}{2}+\frac{1}{2} = 1.$$

Câu 3: Tìm các giới hạn sau:

1). $\lim_{x\to\frac{\pi}{4}}\tan 2x.\tan\left(\frac{\pi}{4}-x\right)$ 2). $\lim_{x\to 0}\frac{\sqrt{1+\tan x}-\sqrt{1+\sin x}}{x^3}$ 3). $\lim_{x\to 1}\frac{\sqrt{x+3}-2}{\tan(x-1)}$

4). $\lim\limits_{x\to -\frac{\pi}{2}} \dfrac{\cos x}{x+\frac{\pi}{2}}$	5). $\lim\limits_{x\to \pi} \dfrac{1+\cos x}{(x-\pi)^2}$	6). $\lim\limits_{x\to 1} \dfrac{\sin(x-1)}{x^2-4x+3}$
7). $\lim\limits_{x\to \frac{\pi}{6}} \dfrac{2\sin x-1}{4\cos^2 x-3}$	8). $\lim\limits_{x\to \frac{\pi}{4}} \dfrac{\sqrt{2}\sin x-1}{2\cos^2 x-1}$	9). $\lim\limits_{x\to \frac{\pi}{6}} \dfrac{\sin\left(\frac{\pi}{6}-x\right)}{1-2\sin x}$

LỜI GIẢI

1). $L=\lim\limits_{x\to \frac{\pi}{4}} \tan 2x.\tan\left(\frac{\pi}{4}-x\right)$. Đặt $t=x-\frac{\pi}{4}$, vì $x\to\frac{\pi}{4}\Rightarrow t\to 0$

$$L=\lim_{t\to 0}\left[\tan\left(2t+\frac{\pi}{2}\right)(-1)\tan t\right]=\lim_{t\to 0}\left(\cot 2t.\tan t\right)$$

$$=\lim_{t\to 0}\frac{\cos 2t}{\sin 2t}\frac{\sin t}{\cos t}=\lim_{t\to 0}\frac{\cos 2t}{2\sin t\cos t}\frac{\sin t}{\cos t}=\lim_{t\to 0}\frac{\cos 2t}{2\cos^2 t}=\frac{1}{2}$$

2). $$\lim_{x\to 0}\frac{\sqrt{1+\tan x}-\sqrt{1+\sin x}}{x^3}=\lim_{x\to 0}\frac{\tan x-\sin x}{x^3\left(\underbrace{\sqrt{1+\tan x}+\sqrt{1+\sin x}}_{A}\right)}=\lim_{x\to 0}\frac{\sin x\left(x-\cos x\right)}{x^3.A.\cos x}$$

$$=\lim_{x\to 0}\frac{2\sin x\sin^2\frac{x}{2}}{x^3.A.\cos x}=\lim_{x\to 0}\left(\frac{\sin x}{x}\right).\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2.\frac{1}{2A.\cos x}=\frac{1}{4}.$$

3). $$\lim_{x\to 1}\frac{\sqrt{x+3}-2}{\tan(x-1)}=\lim_{x\to 1}\frac{x+3-4}{\left(\sqrt{x+3}+2\right)\tan(x-1)}=\lim_{x\to 1}\frac{x-1}{\tan(x-1)}\frac{1}{\sqrt{x+3}+2}$$

(Vì $\lim\limits_{x\to 1}\dfrac{x-1}{\tan(x-1)}=1$, $\lim\limits_{x\to 1}\dfrac{1}{\sqrt{x+3}+2}=\dfrac{1}{4}$)

Vậy $L=\frac{1}{4}$.

4). $L=\lim\limits_{x\to -\frac{\pi}{2}}\dfrac{\cos x}{x+\frac{\pi}{2}}$. Đặt $t=x+\frac{\pi}{2}$, vì $x\to -\frac{\pi}{2}\Rightarrow t\to 0$

$$L=\lim_{t\to 0}\frac{\cos\left(t-\frac{\pi}{2}\right)}{t}=\lim_{t\to 0}\frac{\sin t}{t}=1.$$

5). $L=\lim_{x\to\pi}\frac{1+\cos x}{(x-\pi)^2}$. Đặt $t=x-\pi$, vì $x\to\pi\Rightarrow t\to 0$

$$L=\lim_{t\to 0}\frac{1+\cos(t+\pi)}{t^2}=\lim_{t\to 0}\frac{1-\cos t}{t^2}=\lim_{t\to 0}\frac{2\sin^2\frac{t}{2}}{t^2}=\lim_{t\to 0}\frac{1}{2}\left(\frac{\sin\frac{t}{2}}{\frac{t}{2}}\right)^2=\frac{1}{2}.$$

6). $L=\lim_{x\to 1}\frac{\sin(x-1)}{x^2-4x+3}=\lim_{x\to 1}\frac{\sin(x-1)}{(x-1)(x-3)}$. Đặt $t=x-1$, vì $x\to 1\Rightarrow t\to 0$

$$L=\lim_{t\to 0}\frac{\sin t}{t.(t-2)}=\lim_{t\to 0}\frac{\sin t}{t}\cdot\frac{1}{t-2}=-\frac{1}{2}.$$

7). $L=\lim_{x\to\frac{\pi}{6}}\frac{2\sin x-1}{4\cos^2 x-3}=\lim_{x\to\frac{\pi}{6}}\frac{2\sin x-1}{4\left(1-\sin^2 x\right)-3}=\lim_{x\to\frac{\pi}{6}}\frac{2\sin x-1}{1-4\sin^2 x}$

$$=\lim_{x\to\frac{\pi}{6}}\frac{2\sin x-1}{(1-2\sin x)(1+2\sin x)}=\lim_{x\to\frac{\pi}{6}}\frac{-1}{1+2\sin x}=-\frac{1}{2}$$

8). $L=\lim_{x\to\frac{\pi}{4}}\frac{\sqrt{2}\sin x-1}{2\cos^2 x-1}=\lim_{x\to\frac{\pi}{4}}\frac{\sqrt{2}\sin x-1}{2\left(1-\sin^2 x\right)-1}=\lim_{x\to\frac{\pi}{4}}\frac{\sqrt{2}\sin x-1}{1-2\sin^2 x}$

$$=\lim_{x\to\frac{\pi}{4}}\frac{\sqrt{2}\sin x-1}{\left(1-\sqrt{2}\sin x\right)\left(1+\sqrt{2}\sin x\right)}=\lim_{x\to\frac{\pi}{4}}\frac{-1}{1+\sqrt{2}\sin x}=-\frac{1}{2}.$$

9). $=\lim_{x\to\frac{\pi}{6}}\frac{\sin\left(\frac{\pi}{6}-x\right)}{1-2\sin x}=\lim_{x\to\frac{\pi}{6}}\frac{\sin\left(\frac{\pi}{6}-x\right)}{-2\left(\sin x-\frac{1}{2}\right)}=\lim_{x\to\frac{\pi}{6}}\frac{\sin\left(x-\frac{\pi}{6}\right)}{2\left(\sin x-\sin\frac{\pi}{6}\right)}$

$$=\lim_{x\to\frac{\pi}{6}}\frac{2\sin\left(\frac{x}{2}-\frac{\pi}{12}\right)\cos\left(\frac{x}{2}-\frac{\pi}{12}\right)}{4\cos\left(\frac{x}{2}+\frac{\pi}{12}\right)\sin\left(\frac{x}{2}-\frac{\pi}{12}\right)}=\lim_{x\to\frac{\pi}{6}}\frac{1}{2}\frac{\cos\left(\frac{x}{2}-\frac{\pi}{12}\right)}{\cos\left(\frac{x}{2}+\frac{\pi}{12}\right)}=\frac{\sqrt{3}}{3}$$

Câu 4: Tìm các giới hạn sau:

1). $\lim\limits_{x\to\frac{\pi}{6}}\dfrac{1-2\sin x}{\dfrac{\pi}{6}-x}$ 2). $\lim\limits_{x\to\frac{\pi}{4}}\dfrac{\sin\left(\dfrac{\pi}{4}-x\right)}{1-\sqrt{2}\sin x}$ 3). $\lim\limits_{x\to\frac{\pi}{4}}\dfrac{\sqrt{2}-2\cos x}{\sin\left(x-\dfrac{\pi}{4}\right)}$

4). $\lim\limits_{x\to 0}\dfrac{\sqrt{x^2+1}-\cos 2x}{x^2}$ 5). $\lim\limits_{x\to 0}\dfrac{\sqrt{1+2x}-\cos x-x}{x^2}$ 6). $\lim\limits_{x\to 0}\dfrac{\sqrt[3]{2x+1}-\sqrt{1-x}}{\sin 2x}$

7). $\lim\limits_{x\to\frac{\pi}{3}}\dfrac{\sin x-\sqrt{3}\cos x}{\sin 3x}$ 8). $\lim\limits_{x\to 0}\dfrac{1-\cos x\sqrt{\cos 2x}}{x^2}$ 9). $\lim\limits_{x\to 0}\dfrac{1-\sqrt[3]{\cos x}}{\tan^2 x}$

LỜI GIẢI

1). $\lim\limits_{x\to\frac{\pi}{6}}\dfrac{1-2\sin x}{\dfrac{\pi}{6}-x}=\lim\limits_{x\to\frac{\pi}{6}}\dfrac{2\sin x-1}{x-\dfrac{\pi}{6}}=\lim\limits_{x\to\frac{\pi}{6}}\dfrac{2\left(\sin x-\dfrac{1}{2}\right)}{x-\dfrac{\pi}{6}}=\lim\limits_{x\to\frac{\pi}{6}}\dfrac{2\left(\sin x-\sin\dfrac{\pi}{6}\right)}{x-\dfrac{\pi}{6}}$

$=\lim\limits_{x\to\frac{\pi}{6}}\dfrac{4\cos\left(\dfrac{x}{2}+\dfrac{\pi}{12}\right)\sin\left(\dfrac{x}{2}-\dfrac{\pi}{12}\right)}{2\left(\dfrac{x}{2}-\dfrac{\pi}{12}\right)}=\lim\limits_{x\to\frac{\pi}{6}}\dfrac{\sin\left(\dfrac{x}{2}-\dfrac{\pi}{12}\right)}{\left(\dfrac{x}{2}-\dfrac{\pi}{12}\right)}2\cos\left(\dfrac{x}{2}+\dfrac{\pi}{12}\right)=\sqrt{3}$

2). $\lim\limits_{x\to\frac{\pi}{4}}\dfrac{\sin\left(\dfrac{\pi}{4}-x\right)}{1-\sqrt{2}\sin x}=\lim\limits_{x\to\frac{\pi}{4}}\dfrac{\sin\left(x-\dfrac{\pi}{4}\right)}{\sqrt{2}\sin x-1}=\lim\limits_{x\to\frac{\pi}{4}}\dfrac{\sin\left(x-\dfrac{\pi}{4}\right)}{\sqrt{2}\left(\sin x-\dfrac{\sqrt{2}}{2}\right)}=\lim\limits_{x\to\frac{\pi}{4}}\dfrac{\sin\left(x-\dfrac{\pi}{4}\right)}{\sqrt{2}\left(\sin x-\sin\dfrac{\pi}{4}\right)}$

$=\lim\limits_{x\to\frac{\pi}{4}}\dfrac{2\sin\left(\dfrac{x}{2}-\dfrac{\pi}{8}\right)\cos\left(\dfrac{x}{2}-\dfrac{\pi}{8}\right)}{\sqrt{2}\cos\left(\dfrac{x}{2}+\dfrac{\pi}{8}\right)\sin\left(\dfrac{x}{2}-\dfrac{\pi}{8}\right)}=\lim\limits_{x\to\frac{\pi}{4}}\dfrac{\sqrt{2}\cos\left(\dfrac{x}{2}-\dfrac{\pi}{8}\right)}{\cos\left(\dfrac{x}{2}+\dfrac{\pi}{8}\right)}=2$

3). $\lim\limits_{x\to\frac{\pi}{4}}\dfrac{\sqrt{2}-2\cos x}{\sin\left(x-\dfrac{\pi}{4}\right)}=\lim\limits_{x\to\frac{\pi}{4}}\dfrac{-2\left(\cos x-\dfrac{\sqrt{2}}{2}\right)}{\sin\left(x-\dfrac{\pi}{4}\right)}=\lim\limits_{x\to\frac{\pi}{4}}\dfrac{-2\left(\cos x-\cos\dfrac{\pi}{4}\right)}{\sin\left(x-\dfrac{\pi}{4}\right)}$

$=\lim\limits_{x\to\frac{\pi}{4}}\dfrac{4\sin\left(\dfrac{x}{2}+\dfrac{\pi}{8}\right)\sin\left(\dfrac{x}{2}-\dfrac{\pi}{8}\right)}{2\sin\left(\dfrac{x}{2}-\dfrac{\pi}{8}\right)\cos\left(\dfrac{x}{2}-\dfrac{\pi}{8}\right)}=\lim\limits_{x\to\frac{\pi}{4}}\dfrac{2\sin\left(\dfrac{x}{2}+\dfrac{\pi}{8}\right)}{\cos\left(\dfrac{x}{2}-\dfrac{\pi}{8}\right)}=\sqrt{2}$