$(*) \Leftrightarrow \left(\sqrt{2x-x^2}\right)^3 . \dfrac{-1}{\left(\sqrt{2x-x^2}\right)^3} + 1 = 0 \Leftrightarrow -1+1=0$ (đpcm).

c). Cho hàm số: $y = x\tan x$ chứng minh: $x^2.y'' - 2\left(x^2+y^2\right)(1+y) = 0 \quad (*)$

Ta có: $y' = \left(x\tan x\right)' = x'.\tan x + x.\left(\tan x\right)' = \tan x + x\left(1+\tan^2 x\right)$

$y'' = \left(\tan x\right)' + x'.\left(1+\tan x\right) + x.\left(1+\tan x\right)' = 2\left(1+\tan^2 x\right) + x.\left(2\tan x\right).\left(\tan^2+1\right)$

$= 2\left(1+\tan^2 x\right)\left(1+x\tan x\right)$

$(*) \Leftrightarrow 2x^2\left(1+\tan^2 x\right).\left(1+x\tan x\right) - 2\left(x^2+x^2\tan^2 x\right)\left(1+x\tan x\right) = 0$

$\Leftrightarrow 2x^2\left(1+\tan^2 x\right)\left(1+x\tan x\right) - 2x^2\left(1+\tan^2 x\right)\left(1+x\tan x\right) = 0$

$\Leftrightarrow 0 = 0$ (đpcm).

d). Cho hàm số: $y = \dfrac{x-3}{x+4}$ chứng minh: $2\left(y'\right)^2 = (y-1).y'' \quad (*)$

Ta có: $y' = \left(\dfrac{x-3}{x+4}\right)' = \dfrac{7}{\left(x+4\right)^2}$

$y'' = \dfrac{-7\left(\left(x+4\right)^2\right)'}{\left(x+4\right)^4} = \dfrac{-14}{\left(x+4\right)^3}$

$(*) \Leftrightarrow 2\left(\dfrac{7}{\left(x+4\right)^2}\right)^2 = \left(\dfrac{x-3}{x+4}-1\right).\left(\dfrac{-14}{\left(x+4\right)^3}\right) \Leftrightarrow \dfrac{98}{\left(x+4\right)^4} = \dfrac{98}{\left(x+4\right)^4}$ (đpcm).

e) Cho hàm số $y = \cos^2 3x$ chứng minh: $18\left(2y-1\right) + y'' = 0 \quad (*)$

Ta có: $y = \cos^2 3x$

$y' = 2.\cos 3x\left(\cos 3x\right)' = 2\cos 3x.\left(-\sin 3x\right)\left(3x\right)' = -3\sin 6x$

$y'' = -18\cos 6x$

$(*) \Leftrightarrow 18\left(2\cos^2 3x - 1\right) - 18\cos 6x = 0 \Leftrightarrow 18.\cos 6x - 18\cos 6x = 0$ (đpcm).

Bài 12:

a).Cho hàm số $y = \dfrac{\sin^3 x + \cos^3 x}{1-\sin x.\cos x}$. Chứng minh $y''+y = 0 \quad (*)$

b). Cho hàm số $y = \left(x^2-1\right)^2$. Chứng minh: $y^4 + 2xy''' - 4y'' = 40 \quad (*)$

c). Cho hàm số $y=\sqrt{x+\sqrt{1+x^2}}$. Chứng minh: $4\left(x^2+1\right).y''+4x.y'-y=0\ \ (*)$

d). Chứng minh $\left(1+x^2\right).y''+x.y'-k^2.y=0$ nếu $y=\left(x+\sqrt{x^2+1}\right)^k$

LỜI GIẢI

a).Cho hàm số $y=\dfrac{\sin^3 x+\cos^3 x}{1-\sin x.\cos x}$ chứng minh $y''+y=0\ \ (*)$

Ta có: $y=\dfrac{\left(\sin x+\cos x\right)\left(\sin^2 x+\cos^2 x-\sin x\cos x\right)}{1-\sin x\cos x}$

$=\dfrac{\left(\sin x+\cos x\right)\left(1-\sin x\cos x\right)}{1-\sin x\cos x}=\sin x+\cos x$

$y'=\cos x-\sin x$

$y''=-\sin x-\cos x$

$(*)\Leftrightarrow -\sin x-\cos x+\sin x+\cos x=0\Leftrightarrow 0=0$ (đpcm).

b). Cho hàm số $y=\left(x^2-1\right)^2$. Chứng minh: $y^4+2xy'''-4y''=40\ \ (*)$

Ta có: $y=x^4-2x^2+1$

$y'=4x^3-4x$

$y''=12x^2-4$

$y'''=24x$

$y''''=24.$

$(*)\Leftrightarrow 24+2x\left(24x\right)-4\left(12x^2-4\right)=40.$

$\Leftrightarrow 24+48x^2-48x^2+16=40\Leftrightarrow 40=40$ (đpcm).

c). Cho hàm số $y=\sqrt{x+\sqrt{1+x^2}}$. Chứng minh: $4\left(x^2+1\right).y''+4x.y'-y=0\ \ (*)$

Ta có: $y'=\dfrac{1}{2\sqrt{x+\sqrt{1+x^2}}}.\left(1+\dfrac{x}{\sqrt{1+x^2}}\right)=\dfrac{\sqrt{x+\sqrt{1+x^2}}}{2\sqrt{1+x^2}}$

$y''=\dfrac{\left(\sqrt{x+\sqrt{1+x^2}}\right)'.2\sqrt{1+x^2}-\left(2\sqrt{1+x^2}\right)'.\sqrt{x+\sqrt{1+x^2}}}{\left(2\sqrt{1+x^2}\right)^2}$

$$= \frac{\sqrt{x+\sqrt{1+x^2}}.\left(2\sqrt{1+x^2}-4x\right)}{8\left(1+x^2\right).\sqrt{1+x^2}}$$

$$(*) \Leftrightarrow 4\left(x^2+1\right)\frac{\sqrt{x+\sqrt{1+x^2}}\left(2\sqrt{1+x^2}-4x\right)}{8\left(1+x^2\right)\sqrt{1+x^2}}+4x.\frac{\sqrt{x+\sqrt{1+x^2}}}{2\sqrt{1+x^2}}-\sqrt{x+\sqrt{1+x^2}}=0$$

$$\Leftrightarrow \frac{\sqrt{x+\sqrt{1+x^2}}.\left(2\sqrt{1+x^2}-4x\right)}{2\sqrt{1+x^2}}+2x\frac{\sqrt{x+\sqrt{1+x^2}}}{\sqrt{1+x^2}}-\sqrt{x+\sqrt{1+x^2}}=0$$

$$\Leftrightarrow \sqrt{x+\sqrt{1+x^2}}-\frac{2x\sqrt{x+\sqrt{1+x^2}}}{\sqrt{1+x^2}}+\frac{2x\sqrt{x+\sqrt{1+x^2}}}{\sqrt{1+x^2}}-\sqrt{x+\sqrt{1+x^2}}=0$$

$\Leftrightarrow 0=0$ (đpcm).

d). Chứng minh $\left(1+x^2\right).y''+x.y'-k^2.y=0$ nếu $y=\left(x+\sqrt{x^2+1}\right)^k$

Ta có: $y=\left(x+\sqrt{x^2+1}\right)^k \Rightarrow y'=k\left(x+\sqrt{x^2+1}\right)^{k-1}.\left(1+\frac{x}{\sqrt{x^2+1}}\right)$

$$= k\left(x+\sqrt{x^2+1}\right)^{k-1}.\left(\frac{x+\sqrt{x^2+1}}{\sqrt{x^2+1}}\right)=k.\frac{\left(x+\sqrt{x^2+1}\right)^2}{\sqrt{x^2+1}}$$

$$y''=k.\frac{\left[\left(x+\sqrt{x^2+1}\right)^k\right]'.\sqrt{x^2+1}-\left(\sqrt{x^2+1}\right)'.\left(x+\sqrt{x^2+1}\right)^k}{x^2+1}$$

$$= k.\frac{\frac{k\left(x+\sqrt{x^2+1}\right)^k}{\sqrt{x^2+1}}.\sqrt{x^2+1}-\frac{x}{\sqrt{x^2+1}}.\left(x+\sqrt{x^2+1}\right)^k}{\left(x^2+1\right)}$$

$$= \frac{k\left(x+\sqrt{x^2+1}\right)^k\left(k\sqrt{x^2+1}-x\right)}{\left(x^2+1\right)\sqrt{x^2+1}}$$

$$(*) \Leftrightarrow \left(1+x^2\right)\frac{k\left(x+\sqrt{x^2+1}\right)^k.\left(k\sqrt{x^2+1}-x\right)}{\left(x^2+1\right)\sqrt{x^2+1}}+\frac{x.k\left(x+\sqrt{1+x^2}\right)^k}{\sqrt{1+x^2}}$$

$$\Leftrightarrow \frac{k\left(x+\sqrt{x^2+1}\right)^k.\left(k\sqrt{x^2+1}-x\right)}{\sqrt{x^2+1}}+\frac{x.k\left(x^2+\sqrt{1+x^2}\right)^k}{\sqrt{1+x^2}}-k^2\left(x+\sqrt{x^2+1}\right)^k=0$$

Quy đồng đặt thừa số chung được:

$\frac{\left(x+\sqrt{x^2+1}\right)^k}{\sqrt{x^2+1}}\left(k^2\sqrt{x^2+1}-kx+kx-k^2\sqrt{x^2+1}\right)=0\Leftrightarrow 0=0$ (đpcm).

Ví dụ 3: Chứng minh rằng với mọi số nguyên $n\geq 1$ ta có:

a)Nếu $y=\frac{1}{x}$ thì $y^n=(-1)^n.\frac{n!}{x^{n+1}}$.

b) Nếu $y=\cos x$ thì $y^{4n}=\cos x$.

Ví dụ 4: Chứng minh rằng:

a)Nếu $y=\sin ax$ thì $y^{4n}=a^{4n}.\sin ax$ (a là hằng số).

b) Nếu $y=\sin^2 x$ thì $y^{4n}=-2^{4n-1}\cos 2x$.