GIỚI HẠN HÀM SỐ LƯỢNG GIÁC

Dạng 4: $\lim\limits_{x\to 0}\frac{\sin x}{x}=1$

Câu 1: Tìm các giới hạn sau:

1). $\lim\limits_{x\to 0}\frac{\sin 5x}{x}$ 2). $\lim\limits_{x\to 0}\frac{\tan 2x}{3x}$ 3). $\lim\limits_{x\to 0}\frac{1-\cos x}{\sin x}$

4). $\lim\limits_{x\to 0}\frac{1-\cos x}{x^2}$ 5). $\lim\limits_{x\to 0}\frac{\sin 5x.\sin 3x.\sin x}{45x^3}$ 6). $\lim\limits_{x\to 0}\frac{\sin 7x-\sin 5x}{\sin x}$

7). $\lim\limits_{x\to 0}\frac{1-\cos 5x}{1-\cos 3x}$ 8). $\lim\limits_{x\to 0}\frac{1-\cos^2 2x}{x.\sin x}$ 9). $L=\lim\limits_{x\to 0}\frac{x.\sin ax}{1-\cos ax}$

LỜI GIẢI

1). $\lim\limits_{x\to 0}\frac{\sin 5x}{x}=\lim\limits_{x\to 0}\frac{1}{5}\cdot\frac{\sin 5x}{5x}=\frac{1}{5}$

2). $\lim\limits_{x\to 0}\frac{\tan 2x}{3x}=\lim\limits_{x\to 0}\frac{2}{3}\cdot\frac{\tan 2x}{2x}=\frac{2}{3}$

3). $\lim\limits_{x\to 0}\frac{1-\cos x}{\sin x}=\lim\limits_{x\to 0}\frac{2\sin^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}=\lim\limits_{x\to 0}\tan\frac{x}{2}=0$

4). $\lim\limits_{x\to 0}\frac{1-\cos x}{x^2}=\lim\limits_{x\to 0}\frac{2\sin^2\frac{x}{2}}{x^2}=\lim\limits_{x\to 0}\frac{1}{2}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2=\frac{1}{2}$

5). $\lim\limits_{x\to 0}\frac{\sin 5x.\sin 3x.\sin x}{45x^3}=\lim\limits_{x\to 0}\frac{1}{3}\cdot\frac{\sin 5x}{5x}\cdot\frac{\sin 3x}{3x}\cdot\frac{\sin x}{x}=\frac{1}{3}$

6). $\lim\limits_{x\to 0}\frac{\sin 7x-\sin 5x}{\sin x}=\lim\limits_{x\to 0}\frac{2\cos 6x\sin x}{\sin x}=\lim\limits_{x\to 0}2\cos 6x=2$

7). $\lim\limits_{x\to 0}\frac{1-\cos 5x}{1-\cos 3x}=\lim\limits_{x\to 0}\frac{2\sin^2\frac{5x}{2}}{2\sin^2\frac{3x}{2}}=\lim\limits_{x\to 0}\frac{25}{9}\cdot\left(\frac{\sin\frac{5x}{2}}{\frac{5x}{2}}\right)^2\cdot\left(\frac{\frac{3x}{2}}{\sin\frac{3x}{2}}\right)^2=\frac{25}{9}$

(Vì $\lim_{x\to 0}\dfrac{\sin\frac{5x}{2}}{\frac{5x}{2}}=1,\ \lim_{x\to 0}\dfrac{\frac{3x}{2}}{\sin\frac{3x}{2}}=1$)

8). $\lim_{x\to 0}\dfrac{1-\cos^2 2x}{x.\sin x}=\lim_{x\to 0}\dfrac{(1-\cos 2x)(1+\cos 2x)}{x.\sin x}$

$=\lim_{x\to 0}\dfrac{2\sin^2 x(1+\cos 2x)}{x.\sin x}=\lim_{x\to 0}2(1+\cos 2x)\dfrac{\sin x}{x}=4$

9). $L=\lim_{x\to 0}\dfrac{x.\sin ax}{1-\cos ax}=\lim_{x\to 0}\dfrac{x.2\sin\frac{ax}{2}\cos\frac{ax}{2}}{2\sin^2\frac{ax}{2}}=\lim_{x\to 0}\dfrac{x}{\sin\frac{ax}{2}}\cdot\cos\frac{ax}{2}=\lim_{x\to 0}\dfrac{\frac{ax}{2}}{\sin\frac{ax}{2}}\cdot\dfrac{\cos\frac{ax}{2}}{\frac{a}{2}}$

(Vì $\lim_{x\to 0}\dfrac{\frac{ax}{2}}{\sin\frac{ax}{2}}=1$ và $\lim_{x\to 0}\dfrac{\cos\frac{ax}{2}}{\frac{a}{2}}=\dfrac{2}{a}$). Vậy $L=\dfrac{2}{a}$.

Câu 2: Tìm các giới hạn sau:

1). $\lim_{x\to 0}\dfrac{1-\cos ax}{1-\cos bx}$ 2). $\lim_{x\to 0}\dfrac{\sin x.\sin 2x....\sin nx}{n!x^n}$ 3). $\lim_{x\to 0}\dfrac{1-\cos ax}{x^2}\ (a\neq 0)$

4). $\lim_{x\to 0}\dfrac{\sin x-\tan x}{x^3}$ 5). $\lim_{x\to 0}\dfrac{\tan x-\sin x}{\sin^3 x}$ 6). $\lim_{x\to a}\dfrac{\sin x-\sin a}{x-a}$

7). $\lim_{x\to b}\dfrac{\cos x-\cos b}{x-b}$ 8). $\lim_{x\to 0}\dfrac{1-\sqrt{2x+1}}{\sin 2x}$ 9). $\lim_{x\to 0}\dfrac{\cos(a+x)-\cos(a-x)}{x}$

LỜI GIẢI

1). $L=\lim_{x\to 0}\dfrac{1-\cos ax}{1-\cos bx}=\dfrac{2\sin^2\frac{ax}{2}}{2\sin^2\frac{bx}{2}}=\lim_{x\to 0}\left(\dfrac{a}{b}\cdot\dfrac{\sin\frac{ax}{2}}{\frac{ax}{2}}\cdot\dfrac{\frac{bx}{2}}{\sin\frac{bx}{2}}\right)$

Vì $\lim_{x\to 0}\dfrac{\sin\frac{ax}{2}}{\frac{ax}{2}}=1,\ \lim_{x\to 0}\dfrac{\frac{bx}{2}}{\sin\frac{bx}{2}}=1$. Vậy $L=\dfrac{a}{b}$

2). $L=\lim_{x\to 0}\dfrac{\sin x.\sin 2x....\sin nx}{n!x^n}=\lim_{x\to 0}\dfrac{\sin x.\sin 2x....\sin nx}{1.2.3....nx^n}=\lim_{x\to 0}\dfrac{\sin x}{x}\cdot\dfrac{\sin 2x}{2x}\cdots\dfrac{\sin nx}{nx}$

Vì $\lim\limits_{x\to 0}\dfrac{\sin x}{x}=1,\ \lim\limits_{x\to 0}\dfrac{\sin 2x}{2x}=1,\cdots,\ \lim\limits_{x\to 0}\dfrac{\sin nx}{nx}=1$

Vậy $L=1$.

3). $L=\lim\limits_{x\to 0}\dfrac{1-\cos ax}{x^2}=\lim\limits_{x\to 0}\dfrac{2\sin^2\dfrac{ax}{2}}{x^2}=\lim\limits_{x\to 0}\dfrac{a^2}{4}\left(\dfrac{\sin\dfrac{ax}{2}}{\dfrac{ax}{2}}\right)^2$ (vì $\lim\limits_{x\to 0}\dfrac{\sin\dfrac{ax}{2}}{\dfrac{ax}{2}}=1$).

Vậy $L=\dfrac{a^2}{4}$.

4). $L=\lim\limits_{x\to 0}\dfrac{\sin x-\tan x}{x^3}=\lim\limits_{x\to 0}\dfrac{\sin x-\dfrac{\sin x}{\cos x}}{x^3}=\dfrac{\sin x(\cos x-1)}{x^3\cos x}$

$=\lim\limits_{x\to 0}\dfrac{-2\sin^2\dfrac{x}{2}\sin x}{x^3\cos x}=\lim\limits_{x\to 0}\left(\dfrac{\sin\dfrac{x}{2}}{\dfrac{x}{2}}\right)^2\cdot\dfrac{\sin x}{x}\cdot\dfrac{-1}{2\cos x}$

Vì $\lim\limits_{x\to 0}\dfrac{\sin\dfrac{x}{2}}{\dfrac{x}{2}}=1,\ \lim\limits_{x\to 0}\dfrac{\sin x}{x}=1,\ \lim\limits_{x\to 0}\dfrac{-1}{2\cos x}=-\dfrac{1}{2}$.

Vậy $L=-\dfrac{1}{2}$

5). $\lim\limits_{x\to 0}\dfrac{\tan x-\sin x}{\sin^3 x}=\lim\limits_{x\to 0}\dfrac{\dfrac{\sin x}{\cos x}-\sin x}{\sin^3 x}=\lim\limits_{x\to 0}\dfrac{1-\cos x}{\cos x\sin^2 x}=\lim\limits_{x\to 0}\dfrac{2\sin^2\dfrac{x}{2}}{\cos x\sin^2 x}$

$=\lim\limits_{x\to 0}\dfrac{\dfrac{1}{2}\cdot\left(\dfrac{\sin\dfrac{x}{2}}{\dfrac{x}{2}}\right)^2}{\cos x.\left(\dfrac{\sin x}{x}\right)^2}=\dfrac{1}{2}$

6). $\lim\limits_{x\to a}\dfrac{\sin x-\sin a}{x-a}=\lim\limits_{x\to a}\dfrac{2\sin\dfrac{x+a}{2}\sin\dfrac{x-a}{2}}{x-a}=\lim\limits_{x\to a}\sin\dfrac{x+a}{2}\cdot\dfrac{\sin\dfrac{x-a}{2}}{\dfrac{x-a}{2}}=\sin a$

7). $\lim\limits_{x\to b}\dfrac{\cos x-\cos b}{x-b}=\lim\limits_{x\to b}\dfrac{-2\sin\dfrac{x+b}{2}\sin\dfrac{x-b}{2}}{x-b}=\lim\limits_{x\to b}\left(-\sin\dfrac{x+b}{2}\right).\dfrac{\sin\dfrac{x-b}{2}}{\dfrac{x-b}{2}}=-\sin b$

8). $\lim\limits_{x\to 0}\dfrac{1-\sqrt{2x+1}}{\sin 2x}=\lim\limits_{x\to 0}\dfrac{2x}{\sin 2x}\cdot\dfrac{-1}{1+\sqrt{2x+1}}=-\dfrac{1}{2}$

9). $L=\lim\limits_{x\to 0}\dfrac{\cos(a+x)-\cos(a-x)}{x}=\lim\limits_{x\to 0}\dfrac{-2\sin a\sin x}{x}=\lim\limits_{x\to 0}\dfrac{\sin x}{x}.(-2\sin a)$

(Vì $\lim\limits_{x\to 0}\dfrac{\sin x}{x}=1$). Vậy $L=-2\sin a$

Câu 2: Tìm các giới hạn sau:

1). $\lim\limits_{x\to c}\dfrac{\tan x-\tan c}{x-c}$ 2). $\lim\limits_{x\to 0}\dfrac{1-\cos^3 x}{x\sin x}$ 3). $\lim\limits_{x\to a}\dfrac{\sin^2 x-\sin^2 a}{x^2-a^2}$

4). $\lim\limits_{x\to 0}\dfrac{\cos\alpha x-\cos\beta x}{x^2}$ 5). $\lim\limits_{x\to 0}\dfrac{\sin 5x-\sin 3x}{\sin x}$ 6). $\lim\limits_{x\to 1}(1-x)\tan\dfrac{\pi x}{2}$

7). $\lim\limits_{x\to -2}\dfrac{x^3+8}{\tan(x+2)}$ 8). $\lim\limits_{x\to 0}\dfrac{1-\cos x.\cos 2x.\cos 3x}{1-\cos x}$

9). $\lim\limits_{x\to 0}\dfrac{\sin(a+2x)-2\sin(a+x)+\sin a}{x^2}$ 10) $\lim\limits_{x\to 0}\dfrac{\tan(a+2x)-2\tan(a+x)+\tan a}{x^2}$

LỜI GIẢI

1). $\lim\limits_{x\to c}\dfrac{\tan x-\tan c}{x-c}=\lim\limits_{x\to c}\dfrac{\sin(x-c)}{x-c}\cdot\dfrac{1}{\cos x\cos c}=\dfrac{1}{\cos^2 c}$ (vì $\lim\limits_{x\to c}\dfrac{\sin(x-c)}{x-c}=1$).

2). $\lim\limits_{x\to 0}\dfrac{1-\cos^3 x}{x\sin x}=\lim\limits_{x\to 0}\dfrac{(1-\cos x)(1+\cos x+\cos^2 x)}{x\sin x}$

$=\lim\limits_{x\to 0}\dfrac{2\sin^2\dfrac{x}{2}}{x.2\sin\dfrac{x}{2}\cos\dfrac{x}{2}}(1+\cos x+\cos^2 x)=\lim\limits_{x\to 0}\dfrac{\sin\dfrac{x}{2}}{\dfrac{x}{2}}.\dfrac{1+\cos x+\cos^2 x}{2\cos\dfrac{x}{2}}=\dfrac{3}{2}$.

3). $\lim\limits_{x\to a}\dfrac{\sin^2 x-\sin^2 a}{x^2-a^2}=\lim\limits_{x\to a}\dfrac{(\sin x-\sin a)(\sin x+\sin a)}{(x-a)(x+a)}$

$$=\lim_{x\to a}\frac{2\cos\frac{x+a}{2}\sin\frac{x-a}{2}}{2.\frac{x-a}{2}}\cdot\frac{\sin x+\sin a}{x+a}=\lim_{x\to a}\frac{\sin\frac{x-a}{2}}{\frac{x-a}{2}}\cdot\frac{\cos\frac{x+a}{2}\left(\sin x+\sin a\right)}{x+a}$$

$$=\frac{2\cos a.\sin a}{2a}=\frac{\sin 2a}{2a}.$$

4). $\lim_{x\to 0}\frac{\cos\alpha x-\cos\beta x}{x^2}=\lim_{x\to 0}\frac{-2\sin\frac{x(\alpha+\beta)}{2}\cdot\sin\frac{x(\alpha-\beta)}{2}}{x^2}$

$$=\lim_{x\to 0}\frac{\sin\frac{x(\alpha+\beta)}{2}}{\frac{x(\alpha+\beta)}{2}}\cdot\lim_{x\to 0}\frac{\sin\frac{x(\alpha-\beta)}{2}}{\frac{x(\alpha-\beta)}{2}}.\lim_{x\to 0}\frac{(\alpha+\beta)}{2}\frac{(\alpha-\beta)}{2}(-2)=\frac{\beta^2-\alpha^2}{2}.$$

5). $\lim_{x\to 0}\frac{\sin 5x-\sin 3x}{\sin x}=\lim_{x\to 0}\frac{2\cos 4x\sin x}{\sin x}=\lim_{x\to 0}(2\cos 4x)=2$

6). $L=\lim_{x\to 1}\left(1-x\right)\tan\frac{\pi x}{2}$. Đặt $t=x-1$, vì $x\to 1\Rightarrow t\to 0$

$$L=\lim_{t\to 0}(-t)\tan\frac{\pi}{2}\left(t+1\right)=\lim_{t\to 0}(-t)\tan\left(\frac{\pi}{2}+\frac{\pi}{2}t\right)=\lim_{t\to 0}t\cot\frac{\pi}{2}t$$

$$=\lim_{t\to 0}t.\frac{\cos\frac{\pi}{2}t}{\sin\frac{\pi}{2}t}=\lim_{t\to 0}\frac{\frac{\pi}{2}t}{\sin\frac{\pi}{2}t}\cdot\frac{\cos\frac{\pi}{2}t}{\frac{\pi}{2}}=\frac{2}{\pi}$$

7). $\lim_{x\to -2}\frac{x^3+8}{\tan(x+2)}=\lim_{x\to -2}\frac{\left(x+2\right)\left(x^2-2x+4\right)}{\tan(x+2)}=\lim_{x\to -2}\frac{x+2}{\tan(x+2)}\left(x^2-2x+4\right)=12$

(Vì $\lim_{x\to -2}\frac{x+2}{\tan(x+2)}=1$).

8). $\lim_{x\to 0}\frac{1-\cos x.\cos 2x.\cos 3x}{1-\cos x}$

$$\lim_{x\to 0}\frac{\left(1-\cos x.\right)\cos 2x.\cos 3x+\left(1-\cos 2x\right)\cos 3x+\left(1-\cos 3x\right)}{1-\cos x}$$

$$=\lim_{x\to 0}\frac{\left(1-\cos x.\right)\cos 2x.\cos 3x}{1-\cos x}+\lim_{x\to 0}\frac{\left(1-\cos 2x\right)\cos 3x}{1-\cos x}+\lim_{x\to 0}\frac{1-\cos 3x}{1-\cos x}$$

$$=\lim_{x\to 0}\cos 2x.\cos 3x+\lim_{x\to 0}\frac{2\sin^2 x\cos 3x}{2\sin^2\frac{x}{2}}+\lim_{x\to 0}\frac{2\sin^2\frac{3x}{2}}{2\sin^2\frac{x}{2}}$$

$$=1+\lim_{x\to 0}\frac{4\sin^2\frac{x}{2}\cos^2\frac{x}{2}\cos 3x}{\sin^2\frac{x}{2}}+\lim_{x\to 0}9.\frac{\left(\sin\frac{3x}{2}\Big/\frac{3x}{2}\right)^2}{\left(\sin\frac{x}{2}\Big/\frac{x}{2}\right)^2}=1+4+9=14$$

9). $\lim_{x\to 0}\frac{\sin(a+2x)-2\sin(a+x)+\sin a}{x^2}$

$$=\lim_{x\to 0}\frac{\sin(a+2x)-\sin(a+x)+\sin a-\sin(a+x)}{x^2}$$

$$=\lim_{x\to 0}\frac{2\cos\left(a+\frac{3x}{2}\right)\sin\frac{x}{2}-2\cos\left(a+\frac{x}{2}\right)\sin\frac{x}{2}}{x^2}$$

$$=\lim_{x\to 0}\frac{2\sin\frac{x}{2}\left[\cos\left(a+\frac{3x}{2}\right)-\cos\left(a+\frac{x}{2}\right)\right]}{x^2}=\lim_{x\to 0}\frac{-4\sin\frac{x}{2}\sin(a+x)\sin\frac{x}{2}}{x^2}$$

$$=\lim_{x\to 0}(-1)\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2\sin(a+x)=-\sin a$$

10). $\lim_{x\to 0}\frac{\tan(a+2x)-2\tan(a+x)+\tan a}{x^2}$

$$=\lim_{x\to 0}\frac{\tan(a+2x)-\tan(a+x)-\left(\tan(a+x)-\tan a\right)}{x^2}$$

$$=\lim_{x\to 0}\frac{\frac{\sin x}{\cos(a+2x)\cos(a+x)}-\frac{\sin x}{\cos(a+x)\cos a}}{x^2}$$

$$=\lim_{x\to 0}\frac{\sin x}{x^2}\left(\frac{\cos a-\cos(a+2x)}{\cos(a+2x)\cos(a+x)\cos a}\right)=\lim_{x\to 0}\frac{\sin x}{x^2}\left(\frac{2\sin x\sin(a+x)}{\cos(a+2x)\cos(a+x)\cos a}\right)$$