DẠNG TOÁN 2: XÁC ĐỊNH GIÁ TRỊ CỦA MỘT BIỂU THỨC LƯỢNG GIÁC CÓ ĐIỀU KIỆN.

1. Các ví dụ minh họa.

Ví dụ 1: Cho $\cos 2x = -\frac{4}{5}$, với $\frac{\pi}{4} < x < \frac{\pi}{2}$. Tính $\sin x, \cos x, \sin\left(x+\frac{\pi}{3}\right), \cos\left(2x-\frac{\pi}{4}\right)$.

Lời giải

Vì $\frac{\pi}{4} < x < \frac{\pi}{2}$ nên $\sin x > 0, \ \cos x > 0$.

Áp dụng công thức hạ bậc, ta có :

$\sin^2 x = \frac{1-\cos 2x}{2} = \frac{9}{10} \Rightarrow \sin x = \frac{3}{\sqrt{10}}$

$\cos^2 x = \frac{1+\cos 2x}{2} = \frac{1}{10} \Rightarrow \cos x = \frac{1}{\sqrt{10}}$

Theo công thức cộng, ta có

$\sin\left(x+\frac{\pi}{3}\right) = \sin x\cos\frac{\pi}{3} + \cos x\sin\frac{\pi}{3} = \frac{3}{\sqrt{10}}.\frac{1}{2} + \frac{1}{\sqrt{10}}.\frac{\sqrt{3}}{2} = \frac{3+\sqrt{3}}{2\sqrt{10}}$

$\cos\left(2x-\frac{\pi}{4}\right) = \cos 2x\sin\frac{\pi}{4} + \cos\frac{\pi}{4}\sin 2x = -\frac{4}{5}.\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}.2.\frac{3}{\sqrt{10}}.\frac{1}{\sqrt{10}} = -\frac{\sqrt{2}}{10}$

Ví dụ 2: Cho $\cos 4\alpha + 2 = 6\sin^2\alpha$ với $\frac{\pi}{2} < \alpha < \pi$. Tính $\tan 2\alpha$.

Lời giải

Ta có $\cos 4\alpha + 2 = 6\sin^2\alpha \Leftrightarrow 2\cos^2 2\alpha - 1 + 2 = 3\ \ 1-\cos 2\alpha$

$\Leftrightarrow 2\cos^2 2\alpha + 3\cos 2\alpha - 2 = 0 \Leftrightarrow \ \ 2\cos 2\alpha - 1 \ \ \ \cos 2\alpha + 2 \ \ = 0 \Leftrightarrow \cos 2\alpha = \frac{1}{2}$ (Vì $\cos 2\alpha + 2 > 0$)

Ta có $1 + \tan^2 2\alpha = \frac{1}{\cos^2 2\alpha} \Rightarrow \tan^2 2\alpha = \frac{1}{\cos^2 2\alpha} - 1 = 3$

Vì $\frac{\pi}{2} < \alpha < \pi \Rightarrow \pi < \alpha < 2\pi$ nên $\sin 2\alpha < 0$. Mặt khác $\cos 2\alpha > 0$ do đó $\tan 2\alpha < 0$

Vậy $\tan 2\alpha = -\sqrt{3}$

Ví dụ 3: Cho $\dfrac{1}{\tan^2\alpha}+\dfrac{1}{\cot^2\alpha}+\dfrac{1}{\sin^2\alpha}+\dfrac{1}{\cos^2\alpha}=7$. Tính $\cos 4\alpha$.

Lời giải

Ta có $\dfrac{1}{\tan^2\alpha}+\dfrac{1}{\cot^2\alpha}+\dfrac{1}{\sin^2\alpha}+\dfrac{1}{\cos^2\alpha}=7$

$\Leftrightarrow \dfrac{\sin^2\alpha+1}{\cos^2\alpha}+\dfrac{\cos^2\alpha+1}{\sin^2\alpha}=7$

$\Leftrightarrow \dfrac{\sin^2\alpha\left(\sin^2\alpha+1\right)+\cos^2\alpha\left(\cos^2\alpha+1\right)}{\sin^2\alpha\cos^2\alpha}=7$

$\Leftrightarrow \sin^4\alpha+\cos^4\alpha+1=7\sin^2\alpha\cos^2\alpha$

$\Leftrightarrow \left(\sin^2\alpha+\cos^2\alpha\right)^2-2\sin^2\alpha\cos^2\alpha+1=7\sin^2\alpha\cos^2\alpha$

$\Leftrightarrow 2=9\sin^2\alpha\cos^2\alpha$

$\Leftrightarrow 8=9\left(2\sin\alpha\cos\alpha\right)^2$

$\Leftrightarrow 8=9\sin^2 2\alpha$

$\Leftrightarrow 16=9\left(1-\cos 4\alpha\right)$

$\Leftrightarrow \cos 4\alpha=-\dfrac{7}{9}$

Vậy $\cos 4\alpha=-\dfrac{7}{9}$

Ví dụ 4: Cho $\sin\alpha+\cos\alpha=\cot\dfrac{\alpha}{2}$ với $0<\alpha<\pi$. Tính $\tan\left(\dfrac{\alpha+2013\pi}{2}\right)$.

Lời giải

Ta có $\sin\alpha=2\sin\dfrac{\alpha}{2}\cos\dfrac{\alpha}{2}=2\cos^2\dfrac{\alpha}{2}.\dfrac{\sin\dfrac{\alpha}{2}}{\cos\dfrac{\alpha}{2}}=\dfrac{2\tan\dfrac{\alpha}{2}}{\tan^2\dfrac{\alpha}{2}+1}$

$\cos\alpha=\cos^2\dfrac{\alpha}{2}-\sin^2\dfrac{\alpha}{2}=\cos^2\dfrac{\alpha}{2}\left(1-\dfrac{\sin^2\dfrac{\alpha}{2}}{\cos^2\dfrac{\alpha}{2}}\right)=\dfrac{1-\tan^2\dfrac{\alpha}{2}}{\tan^2\dfrac{\alpha}{2}+1}$

Do đó $\sin\alpha+\cos\alpha=\cot\frac{\alpha}{2}\Leftrightarrow\frac{2\tan\frac{\alpha}{2}}{\tan^2\frac{\alpha}{2}+1}+\frac{1-\tan^2\frac{\alpha}{2}}{\tan^2\frac{\alpha}{2}+1}=\frac{1}{\tan\frac{\alpha}{2}}$

$\Leftrightarrow\tan\frac{\alpha}{2}\left(1+2\tan\frac{\alpha}{2}-\tan^2\frac{\alpha}{2}\right)=1+\tan^2\frac{\alpha}{2}\Leftrightarrow\tan^3\frac{\alpha}{2}-\tan^2\frac{\alpha}{2}-\tan\frac{\alpha}{2}+1=0$

$\Leftrightarrow\left(\tan\frac{\alpha}{2}-1\right)^2\left(\tan\frac{\alpha}{2}+1\right)=0\Leftrightarrow\tan\frac{\alpha}{2}=\pm1$

Vì $0<\alpha<\pi\Rightarrow0<\frac{\alpha}{2}<\frac{\pi}{2}$ do đó $\tan\frac{\alpha}{2}>0$ nên $\tan\frac{\alpha}{2}=1\Rightarrow\cot\frac{\alpha}{2}=1$

Ta có $\tan\left(\frac{\alpha+2013\pi}{2}\right)=\tan\left(\frac{\alpha}{2}+2006\pi+\frac{\pi}{2}\right)=-\cot\frac{\alpha}{2}=-1$

Vậy $\tan\left(\frac{\alpha+2013\pi}{2}\right)=-1$

***Lưu ý*:** Ta có thể biểu diễn $\sin\alpha,\cos\alpha,\tan\alpha,\cot\alpha$ qua $t=\tan\frac{\alpha}{2}$ như sau:

$\sin\alpha=\frac{2t}{1+t^2},\cos\alpha=\frac{1-t^2}{1+t^2},\tan\alpha=\frac{2t}{1-t^2},\cot\alpha=\frac{1-t^2}{2t}$ với α làm các biểu thức có nghĩa.

Ví dụ 5: Cho $\sin\ \alpha+\beta\ =\frac{1}{3}$, $\tan\alpha=-2\tan\beta$.

Tính $A=\sin\left(\alpha+\frac{3\pi}{8}\right)\cos\left(\alpha+\frac{\pi}{8}\right)+\sin\left(\beta-\frac{5\pi}{12}\right)\sin\left(\beta-\frac{\pi}{12}\right)$.

Lời giải

Ta có $\sin\ \alpha+\beta\ =\frac{1}{3}\Leftrightarrow\sin\alpha\cos\beta+\cos\alpha\sin\beta=\frac{1}{3}$ (1)

$\tan\alpha=-2\tan\beta\Leftrightarrow\sin\alpha\cos\beta=-2\sin\beta\cos\alpha$ (2)

Từ (1) và (2) ta được $\begin{cases}\cos\alpha\sin\beta=-\frac{1}{3}\\ \sin\alpha\cos\beta=-\frac{2}{3}\end{cases}\Rightarrow\begin{cases}\cos^2\alpha\sin^2\beta=\frac{1}{9}\\ \sin^2\alpha\cos^2\beta=\frac{4}{9}\end{cases}\Rightarrow\begin{cases}1-\sin^2\alpha\ \ \sin^2\beta=\frac{1}{9}\\ \sin^2\alpha\ \ 1-\sin^2\beta\ =\frac{4}{9}\end{cases}$

$$\Rightarrow \begin{cases} 1-\sin^2\alpha \ \ \sin^2\beta = \dfrac{1}{9} \\ \sin^2\alpha - \sin^2\beta = \dfrac{1}{3} \end{cases} \Rightarrow \left(1-\sin^2\beta - \frac{1}{3}\right)\sin^2\beta = \frac{1}{9}$$

$$\Rightarrow \sin^4\beta - \frac{2}{3}\sin^2\beta + \frac{1}{9} = 0 \Rightarrow \left(\sin^2\beta - \frac{1}{3}\right)^2 = 0 \Rightarrow \sin^2\beta = \frac{1}{3}$$

Do đó $\sin^2\alpha = \sin^2\beta + \dfrac{1}{3} = \dfrac{2}{3}$

Ta có $\sin\left(\alpha + \dfrac{3\pi}{8}\right)\cos\left(\alpha + \dfrac{\pi}{8}\right) = \dfrac{1}{2}\left[\sin\left(2\alpha + \dfrac{\pi}{2}\right) - \sin\dfrac{\pi}{4}\right] = \dfrac{1}{2}\left(\cos 2\alpha - \dfrac{\sqrt{2}}{2}\right)$

$$= \frac{1}{2}\left(1 - 2\sin^2\alpha - \frac{\sqrt{2}}{2}\right) = \frac{1}{2}\left(1 - 2.\frac{2}{3} - \frac{\sqrt{2}}{2}\right) = -\frac{2+3\sqrt{2}}{12}$$

$$\sin\left(\beta - \frac{\pi}{12}\right)\cos\left(\beta - \frac{5\pi}{12}\right) = \frac{1}{2}\left[\sin\left(2\beta - \frac{\pi}{2}\right) + \sin\frac{\pi}{3}\right] = \frac{1}{2}\left[-\cos 2\beta + \frac{\sqrt{3}}{2}\right]$$

$$= \frac{1}{2}\left(-1 + 2\sin^2\beta + \frac{\sqrt{3}}{2}\right) = \frac{1}{2}\left(-1 + 2.\frac{1}{3} + \frac{\sqrt{3}}{2}\right) = \frac{-2+3\sqrt{2}}{12}$$

Do đó $A = -\dfrac{2+3\sqrt{2}}{12} + \dfrac{-2+3\sqrt{2}}{12} = -\dfrac{1}{3}$

2. Bài tập luyện tập.

Bài 6.35: Cho $\cos 2x = \dfrac{3}{5}$ (với $\dfrac{3\pi}{4} < x < \pi$). Tính $\sin x$, $\cos x$, $\tan\left(x - \dfrac{\pi}{4}\right)$

Bài 6.36: Tính giá trị của biểu thức lượng giác, khi biết:

a) $\sin(a-b)$, $\cos(a+b)$, $\tan(a+b)$ khi $\sin \text{a} = \dfrac{8}{17}$, $\tan b = \dfrac{5}{12}$ và *a, b* là các góc nhọn.

b) $\cos\left(\dfrac{\pi}{3} - \alpha\right)$ *khi* $\sin\alpha = -\dfrac{12}{13}$, $\dfrac{3\pi}{2} < \alpha < 2\pi$

c) $\tan\left(\alpha + \dfrac{\pi}{3}\right)$ *khi* $\sin\alpha = \dfrac{3}{5}$, $\dfrac{\pi}{2} < \alpha < \pi$

Bài 6.37: Cho $2\cos\ \alpha+\beta\ = \cos\alpha\cos\ \pi+\beta$. Tính $A = \dfrac{1}{2\sin^2\alpha + 3\cos^2\alpha} + \dfrac{1}{2\sin^2\beta + 3\cos^2\beta}$.

Bài 6.38: a) Cho $\tan\alpha = \dfrac{m}{n}$. Tính $A = m\sin 2\alpha + n\cos 2\alpha$.

b) Cho $\frac{\cos\left(\alpha+\beta\right)}{\cos\left(\alpha-\beta\right)}=\frac{m}{n}$. Tính $B=\tan\alpha.\tan\beta$.

c) Cho $\tan\left(\alpha+\beta\right)=m$ và $\tan\left(\alpha-\beta\right)=n$. Tính $\tan 2\alpha$.

Bài 6.39: Cho $\sin\alpha+\cos\alpha=\frac{\sqrt{7}}{2}$ và $0<\alpha<\frac{\pi}{4}$. Tính $\tan\frac{2\alpha+2015\pi}{4}$.