**DẠNG 4: Các giới hạn đặc biệt**

Nhắc lại:

- $\underbrace{1+1+\cdots+1}_{n}=n$
- $a^3-b^3=(a-b)\Big(\underbrace{a^2+ab+b^3}_{3\text{ so hang}}\Big)$
- $a^n-b^n=(a-b)\Big(\underbrace{a^{n-1}+a^{n-2}b+a^{n-3}b^2+\cdots+ab^{n-2}+b^{n-1}}_{n\text{ so hang}}\Big)$

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|---|
| Tìm $L=\lim\limits_{x\to 0}\dfrac{\sqrt[n]{1+ax}-1}{x}$ |

**LỜI GIẢI**

Cách giải: Đặt $t=\sqrt[n]{1+ax}\Rightarrow t^n=1+ax\Rightarrow x=\dfrac{t^n-1}{a}$

Ta có khi $x\to 0$ thì $t\to 1$.

Khi đó $L=\lim\limits_{x\to 1}\dfrac{a(t-1)}{t^n-1}$

$$=\lim\limits_{x\to 1}\frac{a(t-1)}{(t-1)(t^{n-1}+t^{n-2}+\cdots+t+1)}=\lim\limits_{x\to 1}\frac{a}{t^{n-1}+t^{n-2}+\cdots+t+1}=\frac{a}{n}$$

Vậy $L=\lim\limits_{x\to 0}\dfrac{\sqrt[n]{1+ax}-1}{x}=\dfrac{a}{n}$

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|---|
| $L=\lim\limits_{x\to 0}\dfrac{\sqrt[n]{1+ax}-\sqrt[m]{1+bx}}{x}$ |

**LỜI GIẢI**

$L=\lim\limits_{x\to 0}\dfrac{\sqrt[n]{1+ax}-1+1-\sqrt[m]{1+bx}}{x}=\lim\limits_{x\to 0}\dfrac{\sqrt[n]{1+ax}-1}{x}-\lim\limits_{x\to 0}\dfrac{\sqrt[m]{1+bx}-1}{x}=\dfrac{a}{n}-\dfrac{b}{m}$ (áp dụng kết quả bài kế trên).

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| $L=\lim\limits_{x\to 0}\dfrac{\sqrt[n]{1+ax}-1}{\sqrt[m]{1+bx}-1}\quad (ab\neq 0)$ |

**LỜI GIẢI**

$L=\lim\limits_{x\to 0}\dfrac{\sqrt[n]{1+ax}-1}{x}\cdot\dfrac{x}{\sqrt[m]{1+bx}-1}=\dfrac{a}{n}\cdot\dfrac{m}{b}=\dfrac{am}{bn}$ (áp dụng kết quả bài trên).

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|---|
| $L=\lim\limits_{x\to 0}\dfrac{\sqrt[n]{1+ax}-\sqrt[m]{1+bx}}{\sqrt{1+x}-1}$ |

**LỜI GIẢI**

$L=\lim\limits_{x\to 0}\dfrac{\sqrt[n]{1+ax}-1+1-\sqrt[m]{1+bx}}{x}\cdot\dfrac{x}{\sqrt{1+x}-1}=\lim\limits_{x\to 0}\left(\dfrac{\sqrt[n]{1+ax}-1}{x}-\dfrac{\sqrt[m]{1+bx}-1}{x}\right)\left(\sqrt{1+x}+1\right)$

$=2\left(\dfrac{a}{n}-\dfrac{b}{m}\right)$

| |
|---|
| $L=\lim\limits_{x\to 1}\dfrac{\sqrt[m]{x}-1}{\sqrt[n]{x}-1}$ |

**LỜI GIẢI**

Đặt $t=\sqrt[mn]{x}\Rightarrow x=t^{mn}$, vậy $\sqrt[m]{x}=t^n,\sqrt[n]{x}=t^m$

$$L=\lim_{t\to 1}\frac{t^n-1}{t^m-1}=\lim_{t\to 1}\frac{(t-1)(t^{n-1}+t^{n-2}+\cdots+t+1)}{(t-1)(t^{m-1}+t^{m-2}+\cdots+t+1)}\lim_{t\to 1}\frac{t^{n-1}+t^{n-2}+\cdots+t+1}{t^{m-1}+t^{m-2}+\cdots+t+1}=\frac{n}{m}.$$

$$L=\lim_{x\to 1}\frac{x+x^2+x^3+\cdots+x^n-n}{x+x^2+x^3+\cdots+x^m-m}$$

**LỜI GIẢI**

Ta có: $x+x^2+x^3+\cdots+x^n-n=(x-1)+(x^2-1)+(x^3-1)+\cdots+(x^n-1)$

$=(x-1)+(x-1)(x+1)+(x-1)(x^2+x+1)+\cdots+(x-1)(x^{n-1}+x^{n-1}+\cdots+1)$

$=(x-1)\left[1+(x+1)+(x^2+x+1)+\cdots+(x^{n-1}+x^{n-1}+\cdots+1)\right]$

Tương tự: : $x+x^2+x^3+\cdots+x^m-m=(x-1)+(x^2-1)+(x^3-1)+\cdots+(x^m-1)$

$=(x-1)+(x-1)(x+1)+(x-1)(x^2+x+1)+\cdots+(x-1)(x^{m-1}+x^{m-1}+\cdots+1)$

$=(x-1)\left[1+(x+1)+(x^2+x+1)+\cdots+(x^{m-1}+x^{m-1}+\cdots+1)\right]$

Vậy $L=\lim_{x\to 1}\frac{(x-1)\left[1+(x+1)+(x^2+x+1)+\cdots+(x^{n-1}+x^{n-1}+\cdots+1)\right]}{(x-1)\left[1+(x+1)+(x^2+x+1)+\cdots+(x^{m-1}+x^{m-1}+\cdots+1)\right]}$

$$=\lim_{x\to 1}\frac{1+(x+1)+(x^2+x+1)+\cdots+(x^{n-1}+x^{n-1}+\cdots+1)}{1+(x+1)+(x^2+x+1)+\cdots+(x^{m-1}+x^{m-1}+\cdots+1)}$$

$$=\frac{1+2+3+\cdots+n}{1+2+3+\cdots+m}=\frac{\frac{n(n+1)}{2}}{\frac{m(m+1)}{2}}=\frac{n(n+1)}{m(m+1)}.$$

$$L=\lim_{x\to 1}\frac{x^{100}-2x+1}{x^{50}-2x+1}$$

**LỜI GIẢI**

$$L=\lim_{x\to 1}\frac{x^{100}-x-x+1}{x^{50}-x-x+1}=\lim_{x\to 1}\frac{(x^{100}-x)-(x-1)}{(x^{50}-x)-(x-1)}=\lim_{x\to 1}\frac{x(x^{99}-1)-(x-1)}{x(x^{49}-1)-(x-1)}$$

$$=\lim_{x\to 1}\frac{x(x-1)(x^{98}+x^{97}+\cdots+x+1)-(x-1)}{x(x-1)(x^{48}+x^{47}+\cdots+x+1)-(x-1)}=\lim_{x\to 1}\frac{(x-1)(x^{99}+x^{98}+\cdots+x^2+x-1)}{(x-1)(x^{49}+x^{48}+\cdots+x^2+x-1)}$$

$$=\lim_{x\to 1}\frac{x^{99}+x^{98}+\cdots+x^2+x-1}{x^{49}+x^{48}+\cdots+x^2+x-1}=\frac{98}{48}=\frac{49}{24}$$

$$L=\lim_{x\to 1}\frac{x^{n+1}-(n+1)x+n}{(x-1)^2}$$

LỜI GIẢI

Ta có $x^{n+1}-(n+1)x+n=(x^{n+1}-x)-(nx-n)=x(x^n-1)-n(x-1)$

$=x(x-1)(x^{n-1}+x^{n-1}+\cdots+x+1)-n(x-1)=(x-1)(x^n+x^{n-1}+\cdots+x^2+x-n)$

$$=(x-1)\left(\underbrace{x^n+x^{n-1}+\cdots+x^2+x}_{n\ \text{so hang}}-\underbrace{1+1+\cdots+1}_{n\ \text{so hang}}\right)$$

$$=(x-1)\left[(x^n-1)+(x^{n-1}-1)+(x^2-1)+(x-1)\right]$$

$$=(x-1)\left[\underbrace{(x-1)\left(\underbrace{x^{n-1}+x^{n-2}+\cdots+1}_{n}\right)+(x-1)\left(\underbrace{x^{n-2}+x^{n-3}+\cdots+1}_{n-1}\right)+\cdots+(x-1)(x+1)+(x-1)}_{n}\right]$$

$$=(x-1)^2\left[\underbrace{\left(\underbrace{x^{n-1}+x^{n-2}+\cdots+1}_{n}\right)+\left(\underbrace{x^{n-2}+x^{n-3}+\cdots+1}_{n-1}\right)+\cdots+(x+1)+1}_{n}\right]$$

Do đó: $L=\lim\limits_{x\to1}\dfrac{(x-1)^2\left[\underbrace{\left(\underbrace{x^{n-1}+x^{n-2}+\cdots+1}_{n}\right)+\left(\underbrace{x^{n-2}+x^{n-3}+\cdots+1}_{n-1}\right)+\cdots+(x+1)+1}_{n}\right]}{(x-1)^2}$

$$L=\lim_{x\to1}\left[\underbrace{\left(\underbrace{x^{n-1}+x^{n-2}+\cdots+1}_{n}\right)+\left(\underbrace{x^{n-2}+x^{n-3}+\cdots+1}_{n-1}\right)+\cdots+(x+1)+1}_{n}\right]$$

$$=n+(n-1)+\cdots+2+1=\frac{n(n+1)}{2}$$

$$\lim_{x\to1}\left(\frac{m}{1-x^m}-\frac{n}{1-x^n}\right),\ (m,n\in\mathbb{N}^*)$$

**LỜI GIẢI**

$$\lim_{x\to1}\left[\left(\frac{m}{1-x^m}-\frac{1}{1-x}\right)-\left(\frac{n}{1-x^n}-\frac{1}{1-x}\right)\right]=\lim_{x\to1}\left(\frac{m}{1-x^m}-\frac{1}{1-x}\right)-\lim_{x\to1}\left(\frac{n}{1-x^n}-\frac{1}{1-x}\right)$$

$$\lim_{x\to1}\left(\frac{m}{1-x^m}-\frac{1}{1-x}\right)=\lim_{x\to1}\frac{m-\left(1+x+x^2+\cdots+x^{m-1}\right)}{1-x^m}$$

$$=\lim_{x\to1}\frac{(1-x)+(1-x^2)+\cdots+(1-x^{m-1})}{1-x^m}$$

$$\lim_{x\to1}\frac{(1-x)\left[1+(1+x)+\cdots+(1+x+x^2+\cdots+x^{m-2})\right]}{(1-x)(1+x+x^2+\cdots+x^{m-1})}$$

$$\lim_{x\to1}\frac{1+(1+x)+\cdots+(1+x+x^2+\cdots+x^{m-2})}{1+x+x^2+\cdots+x^{m-1}}=\frac{1+2+\cdots+m-1}{m}=\frac{m-1}{2}$$

Tương tự $\lim\limits_{x\to1}\left(\dfrac{n}{1-x^n}-\dfrac{1}{1-x}\right)=\dfrac{n-1}{2}$

Vậy $\lim\limits_{x\to1}\left(\dfrac{m}{1-x^m}-\dfrac{n}{1-x^n}\right)=\dfrac{m-1}{2}-\dfrac{n-1}{2}=\dfrac{m-n}{2}$

Ví dụ: Tìm các giới hạn sau:

a). $\lim\limits_{x\to 1}\frac{\sqrt{3x+1}\sqrt[3]{2-x}-2}{x-1}$ b). $\lim\limits_{x\to 0}\frac{\sqrt{4+x}\sqrt[3]{8+3x}-4}{x^2+x}$

c). $\lim\limits_{x\to 0}\frac{\sqrt{1.2x+1}\sqrt[3]{2.3x+1}\sqrt[4]{3.4x+1}-1}{x}$

LỜI GIẢI

a). $\lim\limits_{x\to 1}\frac{\sqrt{3x+1}\left(\sqrt[3]{2-x}-1\right)+\sqrt{3x+1}-2}{x-1}=\lim\limits_{x\to 1}\frac{\sqrt{3x+1}\left(\sqrt[3]{2-x}-1\right)}{x-1}+\lim\limits_{x\to 1}\frac{\sqrt{3x+1}-2}{x-1}$

• Tính $\lim\limits_{x\to 1}\frac{\sqrt{3x+1}\left(\sqrt[3]{2-x}-1\right)}{x-1}=\lim\limits_{x\to 1}\frac{\sqrt{3x+1}\left(2-x-1\right)}{\left(x-1\right)\left[\left(\sqrt[3]{2-x}\right)^2+\sqrt[3]{2-x}+1\right]}$

$=\lim\limits_{x\to 1}\frac{-\sqrt{3x+1}}{\left(\sqrt[3]{2-x}\right)^2+\sqrt[3]{2-x}+1}=-\frac{2}{3}$

• Tính $\lim\limits_{x\to 1}\frac{\sqrt{3x+1}-2}{x-1}=\lim\limits_{x\to 1}\frac{3x+1-4}{\left(x-1\right)\left(\sqrt{3x+1}+2\right)}=\lim\limits_{x\to 1}\frac{3}{\sqrt{3x+1}+2}=\frac{3}{4}$.

Vậy $\lim\limits_{x\to 1}\frac{\sqrt{3x+1}\sqrt[3]{2-x}-2}{x-1}=-\frac{2}{3}+\frac{3}{4}=\frac{1}{12}$.

b). $\lim\limits_{x\to 0}\frac{\sqrt[3]{8+3x}\left(\sqrt{4+x}-2\right)+2\sqrt[3]{8+3x}-4}{x^2+x}=\lim\limits_{x\to 0}\frac{\sqrt[3]{8+3x}\left(\sqrt{4+x}-2\right)}{x^2+x}+\lim\limits_{x\to 0}\frac{2\sqrt[3]{8+3x}-4}{x^2+x}$

• Tính $\lim\limits_{x\to 0}\frac{\sqrt[3]{8+3x}\left(\sqrt{4+x}-2\right)}{x^2+x}=\lim\limits_{x\to 0}\frac{\sqrt[3]{8+3x}.x}{x\left(x+1\right)\left(\sqrt{4+x}+2\right)}=\lim\limits_{x\to 0}\frac{\sqrt[3]{8+3x}}{\left(x+1\right)\left(\sqrt{4+x}+2\right)}=\frac{1}{2}$

• Tính $\lim\limits_{x\to 0}\frac{2\sqrt[3]{8+3x}-4}{x^2+x}=2\lim\limits_{x\to 0}\frac{8+3x-8}{x\left(x+1\right)\left[\left(\sqrt[3]{8+3x}\right)^2+2\sqrt[3]{8+3x}+4\right]}$

$=2\lim\limits_{x\to 0}\frac{3}{\left(x+1\right)\left[\left(\sqrt[3]{8+3x}\right)^2+2\sqrt[3]{8+3x}+4\right]}=\frac{1}{2}$

Vậy $\lim\limits_{x\to 0}\frac{\sqrt{4+x}\sqrt[3]{8+3x}-4}{x^2+x}=\frac{1}{2}+\frac{1}{2}=1$

c). $L=\lim\limits_{x\to 0}\frac{\sqrt{1.2x+1}\sqrt[3]{2.3x+1}\sqrt[4]{3.4x+1}-1}{x}=\lim\limits_{x\to 0}\frac{\left(\sqrt{2x+1}-1\right)\sqrt[3]{2.3x+1}\sqrt[4]{3.4x+1}}{x}$

$+\lim\limits_{x\to 0}\frac{\left(\sqrt[3]{2.3x+1}-1\right)\sqrt[4]{3.4x+1}}{x}+\lim\limits_{x\to 0}\frac{\sqrt[4]{3.4x+1}-1}{x}$

Ta chứng minh được $\lim\limits_{x\to 0}\frac{\sqrt[n]{ax+1}-1}{x}=\frac{a}{n}\left(a\neq 0, n\in ¥ *\right)$

**GIỚI HẠN CỦA HÀM SỐ KHI $x\to\infty$**

**DẠNG 1: Tính giới hạn trực tiếp**

**Ví dụ: Tính giới các giới hạn sau:**

a). $\lim\limits_{x\to+\infty}(2x^3-3x)$ b). $\lim\limits_{x\to\pm\infty}\sqrt{x^2-3x+4}$ c). $\lim\limits_{x\to-\infty}\left(\sqrt{2x^2+1}+x\right)$

**LỜI GIẢI**

a). $\lim\limits_{x\to+\infty}(2x^3-3x)=\lim\limits_{x\to+\infty}x^3\left(2-\dfrac{3}{x^2}\right)=\lim\limits_{x\to+\infty}2x^3=+\infty$

b). $\lim\limits_{x\to\pm\infty}\sqrt{x^2-3x+4}=\lim\limits_{x\to\pm\infty}|x|\sqrt{1-\dfrac{3}{x}+\dfrac{4}{x^2}}=\left[\begin{array}{l}\lim\limits_{x\to-\infty}\left[-x\sqrt{1-\dfrac{3}{x}+\dfrac{4}{x^2}}\right]\\ \lim\limits_{x\to+\infty}x\sqrt{1-\dfrac{3}{x}+\dfrac{4}{x^2}}\end{array}\right.=\left[\begin{array}{l}\lim\limits_{x\to-\infty}(-x)\\ \lim\limits_{x\to+\infty}x\end{array}\right.=\left[\begin{array}{l}+\infty\\ +\infty\end{array}\right.$

c). $\lim\limits_{x\to-\infty}\left(\sqrt{2x^2+1}+x\right)=\lim\limits_{x\to-\infty}\left(\sqrt{x^2\left(2+\dfrac{1}{x^2}\right)}+x\right)=\lim\limits_{x\to-\infty}\left(|x|\sqrt{2+\dfrac{1}{x^2}}+x\right)$

$=\lim\limits_{x\to-\infty}\left(-x\sqrt{2+\dfrac{1}{x^2}}+x\right)=\lim\limits_{x\to-\infty}x\left(-\sqrt{2}+1\right)=+\infty$ .

**DẠNG 2:** $\dfrac{\infty}{\infty}$

PHƯƠNG PHÁP GIẢI TOÁN: Chia cả tử và mẫu cho $x^k$ là lũy thừa cao nhất của tử và mẫu (hoặc đặt $x^k$ làm nhân tử chung).

**Ví dụ 1: Tìm giới hạn của các hàm số sau:**

a). $\lim\limits_{x\to-\infty}\dfrac{3x(2x^2-1)}{(5x-1)(x^2+2x)}$ b). $\lim\limits_{x\to+\infty}\sqrt[3]{\dfrac{2x^5+x^3-1}{(2x^2-1)(x^3+x)}}$ c). $\lim\limits_{x\to+\infty}\dfrac{x\sqrt{x}+1}{x^2+x+1}$

d). $L=\lim\limits_{x\to-\infty}\dfrac{2|x|+3}{\sqrt{x^2+x+5}}$ e). $\lim\limits_{x\to-\infty}x.\sqrt{\dfrac{2x^3+x}{x^5-x^2+3}}$ f). $\lim\limits_{x\to+\infty}\dfrac{\sqrt{2x^4+x^2-1}}{1-2x}$.

**LỜI GIẢI**

a). $\lim\limits_{x\to-\infty}\dfrac{3x(2x^2-1)}{(5x-1)(x^2+2x)}=\lim\limits_{x\to-\infty}\dfrac{3x.x^2\left(2-\dfrac{1}{x^2}\right)}{x\left(5-\dfrac{1}{x}\right)x^2\left(1+\dfrac{2}{x}\right)}=\lim\limits_{x\to-\infty}\dfrac{3\left(2-\dfrac{1}{x^2}\right)}{\left(5-\dfrac{1}{x}\right)\left(1+\dfrac{2}{x}\right)}=\dfrac{3.2}{5.1}=\dfrac{6}{5}$.

b). $\lim\limits_{x\to+\infty}\sqrt[3]{\dfrac{2x^5+x^3-1}{(2x^2-1)(x^3+x)}}=\lim\limits_{x\to+\infty}\sqrt[3]{\dfrac{x^5\left(2+\dfrac{1}{x^2}-\dfrac{1}{x^5}\right)}{x^2\left(2-\dfrac{1}{x^2}\right)x^3\left(1+\dfrac{1}{x^2}\right)}}=\lim\limits_{x\to+\infty}\sqrt[3]{\dfrac{\left(2+\dfrac{1}{x^2}-\dfrac{1}{x^5}\right)}{\left(2-\dfrac{1}{x^2}\right)\left(1+\dfrac{1}{x^2}\right)}}=1$

c). $\lim\limits_{x\to+\infty}\dfrac{x\sqrt{x}+1}{x^2+x+1}=\lim\limits_{x\to+\infty}\dfrac{\dfrac{x\sqrt{x}+1}{x^2}}{\dfrac{x^2+x+1}{x^2}}=\lim\limits_{x\to+\infty}\dfrac{\dfrac{1}{\sqrt{x}}+\dfrac{1}{x^2}}{1+\dfrac{1}{x}+\dfrac{1}{x^2}}=\dfrac{0}{1}=0$.

d). $L=\lim\limits_{x\to-\infty}\dfrac{2|x|+3}{\sqrt{x^2+x+5}}$

vì $x\to-\infty\Rightarrow x<0\Rightarrow|x|=-x$. Vậy $L=\lim\limits_{x\to-\infty}\dfrac{-2x+3}{\sqrt{x^2+x+5}}$

$$=\lim_{x\to-\infty}\frac{-2x+3}{\sqrt{x^2\left(1+\frac{1}{x}+\frac{5}{x^2}\right)}}=\lim_{x\to-\infty}\frac{-2x+3}{|x|\sqrt{1+\frac{1}{x}+\frac{5}{x^2}}}=\lim_{x\to-\infty}\frac{x\left(-2+\frac{3}{x}\right)}{-x\sqrt{1+\frac{1}{x}+\frac{5}{x^2}}}=\lim_{x\to-\infty}\frac{-2+\frac{3}{x}}{-\sqrt{1+\frac{1}{x}+\frac{5}{x^2}}}=2$$

e). $$\lim_{x\to-\infty}x.\sqrt{\frac{2x^3+x}{x^5-x^2+3}}=\lim_{x\to-\infty}x.\sqrt{\frac{x^3\left(2+\frac{1}{x^2}\right)}{x^5\left(1-\frac{1}{x^3}+\frac{3}{x^5}\right)}}=\lim_{x\to-\infty}x.\sqrt{\frac{\left(2+\frac{1}{x^2}\right)}{x^2\left(1-\frac{1}{x^3}+\frac{3}{x^5}\right)}}$$

$$=\lim_{x\to-\infty}\frac{x}{|x|}\sqrt{\frac{\left(2+\frac{1}{x^2}\right)}{\left(1-\frac{1}{x^3}+\frac{3}{x^5}\right)}}=\lim_{x\to-\infty}-\sqrt{\frac{\left(2+\frac{1}{x^2}\right)}{\left(1-\frac{1}{x^3}+\frac{3}{x^5}\right)}}=-\sqrt{2}$$

f). $$\lim_{x\to+\infty}\frac{\sqrt{2x^4+x^2-1}}{1-2x}=\lim_{x\to+\infty}\frac{\sqrt{x^4\left(2+\frac{1}{x^2}-\frac{1}{x^4}\right)}}{1-2x}$$

$$=\lim_{x\to+\infty}\frac{x^2\sqrt{2+\frac{1}{x^2}-\frac{1}{x^4}}}{1-2x}=\lim_{x\to+\infty}\frac{x\sqrt{2+\frac{1}{x^2}-\frac{1}{x^4}}}{\frac{1}{x}-2}=\lim_{x\to+\infty}\left(-\frac{\sqrt{2}}{2}x\right)=-\infty$$

**Ví dụ 2: Tính các giới hạn sau:**

a). $\lim_{x\to+\infty}(x+1)\sqrt{\frac{x}{2x^4+x^2+1}}$ b). $\lim_{x\to-\infty}\frac{|x|+\sqrt{x^2+x}}{x+10}$ c). $\lim_{x\to-\infty}\frac{\sqrt{x^2-3x}+2x}{3x-1}$

d). $\lim_{x\to+\infty}\frac{\sqrt{x^2+x+2}+3x+1}{\sqrt{4x^2+1}+1-x}$ e). $\lim_{x\to-\infty}\frac{\sqrt{x^2+x+2}+3x+1}{\sqrt{4x^2+1}+1-x}$

f). $\lim_{x\to-\infty}\frac{\sqrt[3]{(x^3+2x^2)^2}+x\sqrt[3]{x^3+2x^2}+x^2}{3x^2-2x}$

**LỜI GIẢI**

a). $\lim_{x\to+\infty}(x+1)\sqrt{\frac{x}{2x^4+x^2+1}}=\lim_{x\to+\infty}\sqrt{\frac{x(x+1)^2}{2x^4+x^2+1}}=\lim_{x\to+\infty}\sqrt{\frac{\frac{1}{x}+\frac{2}{x^2}+\frac{1}{x^3}}{2+\frac{1}{x^2}+\frac{1}{x^4}}}=0$. (Chú thích:

Vì $x\to+\infty$ nên $x>0\Rightarrow(x+1)>0$ do đó ta được được vào trong dấu căn.

b). $$\lim_{x\to-\infty}\frac{|x|+\sqrt{x^2+x}}{x+10}=\lim_{x\to-\infty}\frac{|x|+\sqrt{x^2\left(1+\frac{1}{x}\right)}}{x+10}=\lim_{x\to-\infty}\frac{|x|+|x|\sqrt{1+\frac{1}{x}}}{x+10}=\lim_{x\to-\infty}\frac{-x-x\sqrt{1+\frac{1}{x}}}{x+10}=\lim_{x\to-\infty}\frac{-1-\sqrt{1+\frac{1}{x}}}{1+\frac{10}{x}}=-2.$$

(Chú giải: Vì $x\to-\infty$ nên $x<0$ do đó $|x|=-x$).

c). $\lim\limits_{x\to-\infty}\dfrac{\sqrt{x^2-3x}+2x}{3x-1}=\lim\limits_{x\to-\infty}\dfrac{\sqrt{x^2\left(1-\frac{3}{x}\right)}+2x}{3x-1}=\lim\limits_{x\to-\infty}\dfrac{|x|\sqrt{1-\frac{3}{x}}+2x}{3x-1}=\lim\limits_{x\to-\infty}\dfrac{-x\sqrt{1-\frac{3}{x}}+2x}{3x-1}=\lim\limits_{x\to-\infty}\dfrac{-\sqrt{1-\frac{3}{x}}+2}{3-\frac{1}{x}}$

$=\dfrac{-1+2}{3}=\dfrac{1}{3}$. (Chú giải: Vì $x\to-\infty$ nên $x<0$ do đó $|x|=-x$).

d). $\lim\limits_{x\to+\infty}\dfrac{\sqrt{x^2+x+2}+3x+1}{\sqrt{4x^2+1}+1-x}=\lim\limits_{x\to+\infty}\dfrac{\frac{\sqrt{x^2+x+2}+3x+1}{x}}{\frac{\sqrt{4x^2+1}+1-x}{x}}=\lim\limits_{x\to+\infty}\dfrac{\frac{\sqrt{x^2+x+2}}{x}+\frac{3x}{x}+\frac{1}{x}}{\frac{\sqrt{4x^2+1}}{x}+\frac{1}{x}-\frac{x}{x}}$

$=\lim\limits_{x\to+\infty}\dfrac{\sqrt{\frac{x^2+x+2}{x^2}}+3+\frac{1}{x}}{\sqrt{\frac{4x^2+1}{x^2}}+\frac{1}{x}-1}=\lim\limits_{x\to+\infty}\dfrac{\sqrt{1+\frac{1}{x}+\frac{2}{x^2}}+3+\frac{1}{x}}{\sqrt{4+\frac{1}{x^2}}+\frac{1}{x}-1}=\dfrac{1+3}{2-1}=4$.

e). $\lim\limits_{x\to-\infty}\dfrac{\sqrt{x^2+x+2}+3x+1}{\sqrt{4x^2+1}+1-x}=\lim\limits_{x\to-\infty}\dfrac{\frac{\sqrt{x^2+x+2}+3x+1}{x}}{\frac{\sqrt{4x^2+1}+1-x}{x}}=\lim\limits_{x\to-\infty}\dfrac{\frac{\sqrt{x^2+x+2}}{x}+\frac{3x}{x}+\frac{1}{x}}{\frac{\sqrt{4x^2+1}}{x}+\frac{1}{x}-\frac{x}{x}}$

$=\lim\limits_{x\to-\infty}\dfrac{-\sqrt{\frac{x^2+x+2}{x^2}}+3+\frac{1}{x}}{-\sqrt{\frac{4x^2+1}{x^2}}+\frac{1}{x}-1}=\lim\limits_{x\to-\infty}\dfrac{-\sqrt{1+\frac{1}{x}+\frac{2}{x^2}}+3+\frac{1}{x}}{-\sqrt{4+\frac{1}{x^2}}+\frac{1}{x}-1}=\dfrac{-1+3}{-2-1}=-\dfrac{2}{3}$

f). $\lim\limits_{x\to-\infty}\dfrac{\sqrt[3]{(x^3+2x^2)^2}+x\sqrt[3]{x^3+2x^2}+x^2}{3x^2-2x}=\lim\limits_{x\to-\infty}\dfrac{\sqrt[3]{\left[x^3\left(1+\frac{2}{x}\right)\right]^2}+\sqrt[3]{x^3\left(1+\frac{2}{x}\right)}+x^2}{3x^2-2x}$

$=\lim\limits_{x\to-\infty}\dfrac{\sqrt[3]{\left(x^2\right)^3\left(1+\frac{2}{x}\right)^2}+\sqrt[3]{x^3\left(1+\frac{2}{x}\right)}+x^2}{3x^2-2x}=\lim\limits_{x\to-\infty}\dfrac{x^2.\sqrt[3]{\left(1+\frac{2}{x}\right)^2}+x^2.\sqrt[3]{1+\frac{2}{x}}+x^2}{3x^2-2x}$

$=\lim\limits_{x\to-\infty}\dfrac{x^2\left[\sqrt[3]{\left(1+\frac{2}{x}\right)^2}+\sqrt[3]{1+\frac{2}{x}}+1\right]}{x^2\left(3-\frac{2}{x}\right)}=\lim\limits_{x\to-\infty}\dfrac{\sqrt[3]{\left(1+\frac{2}{x}\right)^2}+\sqrt[3]{1+\frac{2}{x}}+1}{3-\frac{2}{x}}=\dfrac{1+1+1}{3}=1$.

**Ví dụ 3: Tìm các giới hạn sau:**

a). $\lim\limits_{x\to\infty}x^2\left(\sqrt{\dfrac{x+2}{x}}-\sqrt[3]{\dfrac{x+3}{x}}\right)$ b). $\lim\limits_{x\to-\infty}\left[\sqrt{\dfrac{4x^4+1}{x+2x^4}}-\dfrac{\sqrt{2x^2-4}}{x}\right]$

**LỜI GIẢI**

a). Đặt $x=\dfrac{1}{y}$ khi $x\to\infty$ thì $y\to0$

$$I=\lim_{y\to0}\frac{\sqrt{1+2y}-\sqrt[3]{1+3y}}{y^2}=\lim_{y\to0}\left[\frac{\sqrt{1+2y}-(1+y)}{y^2}-\frac{\sqrt[3]{1+3y}-(1+y)}{y^2}\right]$$

$$=\lim_{y\to0}\left[\frac{-y^2}{y^2\sqrt{1+2y}+(1+y)}+\frac{y^2(y+3)}{y^2(\sqrt[3]{(1+3y)^2}+(1+y)\sqrt[3]{1+3y}+(1+y)^2)}\right]$$

$$=\lim_{y\to0}\left[-\frac{1}{1+y+\sqrt{1+2y}}+\frac{y+3}{(1+y)^2+(1+y)\sqrt[3]{1+3y}+\sqrt[3]{(1+3y)^2}}\right]$$

$=-\frac{1}{2}+1=\frac{1}{2}$. Vậy $I=\frac{1}{2}$

b). $$\lim_{x\to-\infty}\left[\sqrt{\frac{4x^4+1}{x+2x^4}}-\frac{\sqrt{2x^2-4}}{x}\right]=\lim_{x\to-\infty}\left[\sqrt{\frac{\frac{4x^4+1}{x^4}}{\frac{x+2x^4}{x^4}}}-\frac{\sqrt{x^2\left(\frac{2x^2-4}{x^2}\right)}}{x}\right]$$

$$=\lim_{x\to-\infty}\left(\sqrt{\frac{4+\frac{1}{x^4}}{\frac{1}{x^3}+2}}-\frac{|x|\sqrt{2-\frac{4}{x^2}}}{x}\right)=\lim_{x\to-\infty}\left(\sqrt{\frac{4+\frac{1}{x^4}}{\frac{1}{x^3}+2}}+\frac{x\sqrt{2-\frac{4}{x^2}}}{x}\right)=\lim_{x\to-\infty}\left(\sqrt{\frac{4+\frac{1}{x^4}}{\frac{1}{x^3}+2}}+\sqrt{2-\frac{4}{x^2}}\right)$$

$=\sqrt{2}+\sqrt{2}=2\sqrt{2}$.

**DẠNG 3:** $\infty-\infty$

PHƯƠNG PHÁP GIẢI TOÁN: Nhân lượng liên hợp sau đó làm như dạng 1.

**Ví dụ: Tìm các giới hạn sau:**

a). $\lim_{x\to-\infty}(\sqrt{x^2+x}-x)$ b). $\lim_{x\to+\infty}(\sqrt{x^2-3x+2}-x)$ c). $\lim_{x\to-\infty}(\sqrt{x^2+x}+x)$

d). $\lim_{x\to+\infty}(\sqrt{x^2-3x+2}+x)$ e). $\lim_{x\to+\infty}(\sqrt{x+2}-\sqrt{x-2})$

f). $\lim_{x\to+\infty}\left(\sqrt{x^2-4x+3}-\sqrt{x^2-3x+2}\right)$ g). $\lim_{x\to-\infty}\left(\sqrt{x^2-4x+3}-\sqrt{x^2-3x+2}\right)$

**LỜI GIẢI**

a). $$\lim_{x\to-\infty}(\sqrt{x^2+x}-x)=\lim_{x\to-\infty}\left(\sqrt{x^2\left(1+\frac{1}{x}\right)}-x\right)=\lim_{x\to-\infty}\left(|x|\sqrt{1+\frac{1}{x}}-x\right)$$

$=\lim_{x\to-\infty}\left(-x\sqrt{1+\frac{1}{x}}-x\right)=\lim_{x\to-\infty}-x\left(\sqrt{1+\frac{1}{x}}+1\right)=+\infty$. Chú giải: Vì $x\to-\infty$ nên $x<0$ do đó $|x|=-x$.

b). $$\lim_{x\to+\infty}(\sqrt{x^2-3x+2}-x)=\lim_{x\to+\infty}\frac{x^2-3x+2-x^2}{\sqrt{x^2-3x+2}+x}=\lim_{x\to+\infty}\frac{-3x+2}{\sqrt{x^2\left(1-\frac{3}{x}+\frac{2}{x^2}\right)}+x}$$

$$=\lim_{x\to+\infty}\frac{-3x+2}{|x|\sqrt{1-\frac{3}{x}+\frac{2}{x^2}}+x}=\lim_{x\to+\infty}\frac{-3x+2}{x\sqrt{1-\frac{3}{x}+\frac{2}{x^2}}+x}=\lim_{x\to+\infty}\frac{-3+\frac{2}{x}}{\sqrt{1-\frac{3}{x}+\frac{2}{x^2}}+1}=-\frac{3}{2}.$$

Chú thích: Do $x\to+\infty$ nên $x>0$ do đó $|x|=x$

c). $\lim\limits_{x\to-\infty}\left(\sqrt{x^2+x}+x\right)$

$$=\lim_{x\to-\infty}\frac{\left(\sqrt{x^2+x}+x\right)\left(\sqrt{x^2+x}-x\right)}{\sqrt{x^2+x}-x}=\lim_{x\to-\infty}\frac{\left(\sqrt{x^2+x}\right)^2-x^2}{\sqrt{x^2\left(\frac{x^2+x}{x^2}\right)}-x}=\lim_{x\to-\infty}\frac{x}{|x|\sqrt{1+\frac{1}{x}}-x}$$

$$=\lim_{x\to-\infty}\frac{x}{-x\sqrt{1+\frac{1}{x}}-x}=\lim_{x\to-\infty}\frac{x}{-x\left(\sqrt{1+\frac{1}{x}}+1\right)}=\lim_{x\to-\infty}\frac{-1}{\sqrt{1+\frac{1}{x}}+1}=-\frac{1}{2}.$$

Chú giải: Vì $x\to-\infty$ nên $x<0$ do đó $|x|=-x$.

d). $L=\lim\limits_{x\to+\infty}\left(\sqrt{x^2-3x+2}+x\right)=\lim\limits_{x\to+\infty}\left(\sqrt{x^2\left(\frac{x^2-3x+2}{x^2}\right)}+x\right)=\lim\limits_{x\to+\infty}\left(|x|\sqrt{1-\frac{3}{x}+\frac{2}{x^2}}+x\right)$

$$=\lim_{x\to+\infty}\left(x\sqrt{1-\frac{3}{x}+\frac{2}{x^2}}+x\right)=\lim_{x\to+\infty}x\left(\sqrt{1-\frac{3}{x}+\frac{2}{x^2}}+1\right).$$

Do $x\to+\infty$ nên $x>0$ do đó $|x|=x$. Có $\lim\limits_{x\to+\infty}\frac{3}{x}=\lim\limits_{x\to+\infty}\frac{2}{x^2}=0$ nên $\lim\limits_{x\to+\infty}\left(\sqrt{1-\frac{3}{x}+\frac{2}{x^2}}+1\right)=2$ và $\lim\limits_{x\to+\infty}x=+\infty$

. Từ đó suy ra $L=+\infty$.

c). $\lim\limits_{x\to+\infty}(\sqrt{x+2}-\sqrt{x-2})=\lim\limits_{x\to+\infty}\frac{4}{\sqrt{x+2}+\sqrt{x-2}}=\lim\limits_{x\to+\infty}\frac{4}{\sqrt{x}\left(\sqrt{1+\frac{2}{x}}+\sqrt{1-\frac{2}{x}}\right)}=0$

d). $\lim\limits_{x\to+\infty}\left(\sqrt{x^2-4x+3}-\sqrt{x^2-3x+2}\right)=\lim\limits_{x\to+\infty}\frac{x^2-4x+3-(x^2-3x+2)}{\sqrt{x^2-4x+3}+\sqrt{x^2-3x+2}}$

$$\lim_{x\to+\infty}\frac{-x+1}{\sqrt{x^2\left(1-\frac{4}{x}+\frac{3}{x^2}\right)}+\sqrt{x^2\left(1-\frac{3}{x}+\frac{2}{x^2}\right)}}=\lim_{x\to+\infty}\frac{-x\left(1-\frac{1}{x}\right)}{x\left(\sqrt{1-\frac{4}{x}+\frac{3}{x^2}}+\sqrt{1-\frac{3}{x}+\frac{2}{x^2}}\right)}=-\frac{1}{2}.$$

e). $\lim\limits_{x\to-\infty}\left(\sqrt{x^2-4x+3}-\sqrt{x^2-3x+2}\right)=\lim\limits_{x\to-\infty}\frac{x^2-4x+3-(x^2-3x+2)}{\sqrt{x^2-4x+3}+\sqrt{x^2-3x+2}}$

$$\lim_{x\to-\infty}\frac{-x+1}{\sqrt{x^2\left(1-\frac{4}{x}+\frac{3}{x^2}\right)}+\sqrt{x^2\left(1-\frac{3}{x}+\frac{2}{x^2}\right)}}=\lim_{x\to-\infty}\frac{-x\left(1-\frac{1}{x}\right)}{-x\left(\sqrt{1-\frac{4}{x}+\frac{3}{x^2}}+\sqrt{1-\frac{3}{x}+\frac{2}{x^2}}\right)}=\frac{1}{2}.$$

Nhận xét: Nếu $\lim\limits_{x\to-\infty}\left(\sqrt{(ax)^{4k+2}+bx+c}-(ax)^{2k+1}\right)$ hoặc $\lim\limits_{x\to+\infty}\left(\sqrt{(ax)^{2k}+bx+c}+(ax)^{k}\right)$ (với $a>0,\ k\in ¥$ )

ta tính trực tiếp không nhân lượng liên hợp.

**Ví dụ 2: Tìm các giới hạn sau:**

a). $\lim\limits_{x\to-\infty}\left(\sqrt{4x^4+3x^2+1}-2x^2\right)$ b). $\lim\limits_{x\to+\infty}\left(\sqrt[3]{8x^3+1}-2x+1\right)$

**LỜI GIẢI**

a). $\lim\limits_{x\to-\infty}\left(\sqrt{4x^4+3x^2+1}-2x^2\right)=\lim\limits_{x\to-\infty}\dfrac{4x^4+3x^2+1-4x^4}{\sqrt{4x^4+3x^2+1}+2x^2}$

$=\lim\limits_{x\to-\infty}\dfrac{3x^2+1}{\sqrt{x^4\left(4+\dfrac{3}{x^2}+\dfrac{1}{x^4}\right)}+2x^2}=\lim\limits_{x\to-\infty}\dfrac{3+\dfrac{1}{x^2}}{\sqrt{4+\dfrac{3}{x^2}+\dfrac{1}{x^4}}+2}=\dfrac{3}{4}.$

b). $\lim\limits_{x\to+\infty}\left(\sqrt[3]{8x^3+1}-2x+1\right)=\lim\limits_{x\to+\infty}\left(\dfrac{8x^3+1-8x^3}{\left(\sqrt[3]{8x^3+1}\right)^2+\sqrt[3]{8x^3+1}.2x+4x^2}\right)+1$

$=\lim\limits_{x\to+\infty}\left(\dfrac{1}{x^2\sqrt[3]{\left(8+\dfrac{1}{x^3}\right)^2}+x\sqrt[3]{8+\dfrac{1}{x^3}}.2x+4x^2}\right)+1=\lim\limits_{x\to+\infty}\left(\dfrac{1}{x^2\sqrt[3]{\left(8+\dfrac{1}{x^3}\right)^2}+2x^2\sqrt[3]{8+\dfrac{1}{x^3}}+4x^2}\right)+1$

$=\lim\limits_{x\to+\infty}\left(\dfrac{1}{x^2\left(\sqrt[3]{\left(8+\dfrac{1}{x^3}\right)^2}+2\sqrt[3]{\left(8+\dfrac{1}{x^3}\right)}+4\right)}\right)+1=\lim\limits_{x\to+\infty}\left(\dfrac{1}{x^2(4+4+4)}\right)+1$

$=\lim\limits_{x\to+\infty}\left(\dfrac{1}{12x^2}\right)+1=1$

**GIỚI HẠN CỦA HÀM SỐ DẠNG VÔ ĐỊNH** $0.\infty$

PHƯƠNG PHÁP GIẢI TOÁN

Giả sử cần tìm giới hạn của hàm số $h(x)=f(x).g(x)$ khi $x\to x_0$ hoặc $x\to\pm\infty$ trong đó $f(x)\to 0$ và $g(x)\to\pm\infty$. Ta thường biến đổi theo các hướng sau:

Nếu $x\to x_0$ thì ta thường viết $f(x).g(x)=\dfrac{f(x)}{\dfrac{1}{g(x)}}$ sẽ đưa về dạng vô định $\dfrac{0}{0}$.

Nếu $x\to\pm\infty$ thì ta thường viết $f(x).g(x)=\dfrac{g(x)}{\dfrac{1}{f(x)}}$ sẽ đưa về về dạng $\dfrac{\infty}{\infty}$.

Tuy nhiên ở nhiều bài toán giới hạn loại này ta chỉ cần thực hiện một số biến đổi như đưa thừa số vào trong dấu căn thức, quy đồng mẫu số,... ta có thể đưa về giới hạn quen thuộc.

**Ví dụ: Tìm các giới hạn sau:**

a). $\lim\limits_{x\to 3}\left(\frac{1}{x}-\frac{1}{3}\right)\frac{1}{(x-3)^3}$ b). $\lim\limits_{x\to +\infty}(x+2)\sqrt{\frac{x-1}{x^3+x}}$

**LỜI GIẢI**

a). $\lim\limits_{x\to 3}\left(\frac{1}{x}-\frac{1}{3}\right)\frac{1}{(x-3)^3}=\lim\limits_{x\to 3}\frac{3-x}{3x}.\frac{1}{(x-3)^3}=\lim\limits_{x\to 3}-\frac{1}{3x(x-3)^2}=-\infty$.

a). $L=\lim\limits_{x\to(-1)^+}(x^3+1)\sqrt{\frac{x}{x^2-1}}=\lim\limits_{x\to(-1)^+}(x+1)(x^2-x+1)\sqrt{\frac{x}{(x-1)(x+1)}}$

Vì $x\to -1^+\Rightarrow x>-1\Leftrightarrow x+1>0$.

Vậy $L=\lim\limits_{x\to(-1)^+}(x^2-x+1)\sqrt{\frac{(x+1)^2x}{(x-1)(x+1)}}=\lim\limits_{x\to(-1)^+}(x^2-x+1)\sqrt{\frac{(x+1)x}{x-1}}=3.0=0$

b). $\lim\limits_{x\to +\infty}(x+2)\sqrt{\frac{x-1}{x^3+x}}=\lim\limits_{x\to +\infty}(x+2)\sqrt{\frac{x\left(1-\frac{1}{x}\right)}{x^3\left(1+\frac{1}{x^2}\right)}}=\lim\limits_{x\to +\infty}(x+2)\sqrt{\frac{1-\frac{1}{x}}{x^2\left(1+\frac{1}{x^2}\right)}}$

$=\lim\limits_{x\to +\infty}\frac{x+2}{|x|}\sqrt{\frac{1-\frac{1}{x}}{1+\frac{1}{x^2}}}=\lim\limits_{x\to +\infty}\left(1+\frac{2}{x}\right)\sqrt{\frac{1-\frac{1}{x}}{1+\frac{1}{x^2}}}=1$.