BÀI TẬP GIỚI HẠN DÃY SỐ TỔNG HỢP

Câu 1: Tìm các giới hạn sau:

a). $\lim\frac{n-1}{n}$ b). $\lim\frac{n+2}{n+1}$ c). $\lim\frac{n^2-3n+5}{2n^2-1}$

d). $\lim\frac{3n^2+n-5}{2n^2+1}$ e). $\lim\frac{6n^3-2n+1}{2n^3-n}$ f). $\lim\frac{4n^4-n^2+1}{(2n+1)(3-n)(n^2+2)}$.

LỜI GIẢI

a) $\lim\frac{n-1}{n}=\lim\left(1-\frac{1}{n}\right)=0.$

b) $\lim\frac{n+2}{n+1}=\lim\frac{1+\frac{2}{n}}{1+\frac{1}{n}}=1.$ (Chia cả tử và mẫu cho n)

c) Chia cả tử và mẫu cho n^2 được:

$$\lim\frac{n^2-3n+5}{2n^2-1}=\lim\left(\frac{1-\frac{3n}{n^2}+\frac{5}{n^2}}{2-\frac{1}{n^2}}\right)=\lim\left(\frac{1-\frac{3}{n}+\frac{5}{n^2}}{2-\frac{1}{n^2}}\right)=\frac{1}{2}.$$

d) $\lim\frac{3n^2+n-5}{2n^2+1}=\lim\frac{3+\frac{n}{n^2}-\frac{5}{n^2}}{2+\frac{1}{n^2}}=\lim\frac{3+\frac{1}{n}-\frac{5}{n^2}}{2+\frac{1}{n^2}}=\frac{3}{2}.$

e) Chia cả tử và mẫu cho n^3 được:

$\lim\frac{6n^3-2n+1}{2n^3-n}=\lim\frac{6-\frac{2n}{n^3}+\frac{1}{n^3}}{2-\frac{n}{n^3}}=\lim\left(\frac{6-\frac{2}{n^2}+\frac{1}{n^3}}{2-\frac{1}{n^2}}\right)=\frac{6}{2}=3.$ f) $L=\lim\frac{4n^4-n^2+1}{(2n+1)(3-n)(n^2+2)}$

Ta có $4n^4-n^2+1=n^4\left(\frac{4n^4-n^2+1}{n^4}\right)=n^4\left(4-\frac{1}{n^2}+\frac{1}{n^4}\right)$; $2n+1=n\left(\frac{2n+1}{n}\right)=n\left(2+\frac{1}{n}\right)$;

$3-n=n\left(\frac{3-n}{n}\right)=n\left(\frac{3}{n}-1\right)$ và $n^2+2=n^2\left(\frac{n^2+2}{n^2}\right)=n^2\left(1+\frac{2}{n^2}\right)$

Từ đó ta có: $L=\lim\frac{4n^4-n^2+1}{n\left(2+\frac{1}{n}\right)n\left(\frac{3}{n}-1\right)n^2\left(1+\frac{2}{n^2}\right)}$

$$=\lim\frac{n^4\left(4-\frac{1}{n^2}+\frac{1}{n^4}\right)}{n^4\left(2+\frac{1}{n}\right)\left(\frac{3}{n}-1\right)\left(1+\frac{1}{n^2}\right)}=\lim\frac{4-\frac{1}{n^2}+\frac{1}{n^4}}{\left(2+\frac{1}{n}\right)\left(\frac{3}{n}-1\right)\left(1+\frac{1}{n^2}\right)}=\frac{4}{2.1}=2.$$

Câu 2: Tìm các giới hạn sau:

a). $\lim\frac{(n^2+2)(n-1)^2}{(n+1)(2n+3)^2}$ b). $\lim\frac{n^2+2\sqrt{n}+3}{2n^2+n-\sqrt{n}}$

c). $\lim \frac{2n^3-11n+1}{n^2-2}$ d). $\lim \frac{(2n\sqrt{n}+1)(\sqrt{n}+3)}{(n+1)(n+2)}$

LỜI GIẢI

a). $\lim \frac{(n^2+2)(n-1)^2}{(n+1)(2n+3)^2} = \lim \frac{n^2\left(1+\frac{2}{n^2}\right)n^2\left(1-\frac{1}{n}\right)^2}{n\left(1+\frac{1}{n}\right)n^2\left(2+\frac{3}{n}\right)^2} = \lim \frac{\left(1+\frac{2}{n^2}\right)\left(1-\frac{1}{n}\right)^2}{\left(1+\frac{1}{n}\right)\left(2+\frac{3}{n}\right)^2} = \frac{1}{2}.$

b). $\lim \frac{1+\frac{2\sqrt{n}}{n^2}+\frac{3}{n^2}}{2+\frac{n}{n^2}-\frac{\sqrt{n}}{n^2}} = \lim \frac{1+\frac{2}{n\sqrt{n}}+\frac{3}{n^2}}{2+\frac{1}{n}-\frac{1}{n\sqrt{n}}} = \frac{1}{2}.$

c). $\lim \frac{\frac{2n^3}{n^2}-\frac{11n}{n^2}+\frac{1}{n^2}}{\frac{n^2}{n^2}-\frac{2}{n^2}} = \lim \frac{2n-\frac{11}{n}+\frac{1}{n^2}}{1-\frac{2}{n^2}} = \lim 2n = +\infty.$

d). $\lim \frac{(2n\sqrt{n}+1)(\sqrt{n}+3)}{(n+1)(n+2)} = \lim \frac{n\sqrt{n}\left(\frac{2n\sqrt{n}+1}{n\sqrt{n}}\right)\sqrt{n}\left(\frac{\sqrt{n}+3}{\sqrt{n}}\right)}{n\left(\frac{n+1}{n}\right)n\left(\frac{n+2}{n}\right)}$

$= \lim \frac{n\sqrt{n}\left(2+\frac{1}{n\sqrt{n}}\right)\sqrt{n}\left(1+\frac{3}{\sqrt{n}}\right)}{n\left(1+\frac{1}{n}\right)n\left(1+\frac{2}{n}\right)} = \lim \frac{\left(2+\frac{1}{n\sqrt{n}}\right)\left(1+\frac{3}{\sqrt{n}}\right)}{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)} = \frac{2.1}{1.1} = 2.$

Câu 3: Tìm các giới hạn sau:

a). $\lim \frac{\sqrt{9n^2-n+1}}{4n-2}$ b). $\lim \frac{\sqrt{2n^4+3n-2}}{2n^2-n+3}$

c). $\lim \frac{\sqrt{2n+2}-\sqrt{n}}{\sqrt{n}}$ d). $\lim \frac{\sqrt{3n^2+1}-\sqrt{n^2-1}}{n}$

LỜI GIẢI

a) $\lim \frac{\sqrt{9n^2-n+1}}{4n-2} = \lim \frac{\sqrt{n^2\left(9-\frac{1}{n}+\frac{1}{n^2}\right)}}{n\left(4-\frac{2}{n}\right)} = \lim \frac{n\sqrt{9-\frac{1}{n}+\frac{1}{n^2}}}{n\left(4-\frac{2}{n}\right)} = \lim \frac{\sqrt{9-\frac{1}{n}+\frac{1}{n^2}}}{4-\frac{2}{n}} = \frac{3}{2}.$

b). $\lim \frac{\sqrt{2n^4+3n-2}}{2n^2-n+3} = \lim \frac{\sqrt{n^4\left(2+\frac{3}{n^3}-\frac{2}{n^4}\right)}}{n^2\left(2-\frac{1}{n}+\frac{3}{n^2}\right)}$

$= \lim \frac{n^2\sqrt{2+\frac{3}{n^3}-\frac{2}{n^4}}}{n^2\left(2-\frac{1}{n}+\frac{3}{n^2}\right)} = \lim \frac{\sqrt{2+\frac{3}{n^3}-\frac{2}{n^4}}}{2-\frac{1}{n}+\frac{3}{n^2}} = \frac{\sqrt{2}}{2}.$

c). $\lim \dfrac{\sqrt{2n+2}-\sqrt{n}}{\sqrt{n}} = \lim \dfrac{\sqrt{n\left(2+\dfrac{2}{n}\right)}-\sqrt{n}}{\sqrt{n}} = \lim \dfrac{\sqrt{n}\sqrt{2+\dfrac{2}{n}}-\sqrt{n}}{\sqrt{n}} = \lim \dfrac{\sqrt{n}\left(\sqrt{2+\dfrac{2}{n}}-1\right)}{\sqrt{n}}$

$= \lim\left(\sqrt{2+\dfrac{2}{n}}-1\right) = \sqrt{2}-1.$

d). $\lim \dfrac{\sqrt{3n^2+1}-\sqrt{n^2-1}}{n} = \lim \dfrac{\sqrt{n^2\left(3+\dfrac{1}{n^2}\right)}-\sqrt{n^2\left(1-\dfrac{1}{n^2}\right)}}{n}$

$= \lim \dfrac{n\sqrt{3+\dfrac{1}{n^2}}-n\sqrt{1-\dfrac{1}{n^2}}}{n} = \lim \dfrac{n\left(\sqrt{3+\dfrac{1}{n^2}}-\sqrt{1-\dfrac{1}{n^2}}\right)}{n}$

$= \lim\left(\sqrt{3+\dfrac{1}{n^2}}-\sqrt{1-\dfrac{1}{n^2}}\right) = \sqrt{3}-1.$

Câu 4: Tìm các giới hạn sau:

a). $\lim \dfrac{3^n+5.4^n}{4^n+2^n}$ b). $\lim \dfrac{3^n-2.5^n}{7+3.5^n}$ c). $\lim \dfrac{2^n-3^n+5^{n+2}}{2^{n+1}+3^{n+2}+5^{n+1}}$ d). $\lim \dfrac{4.3^n+5^{n+1}}{3.2^n+5^n}$

LỜI GIẢI

a). $\lim \dfrac{3^n+5.4^n}{4^n+2^n} = \lim \dfrac{\dfrac{3^n}{4^n}+\dfrac{5.4^n}{4^n}}{\dfrac{4^n}{4^n}+\dfrac{2^n}{4^n}} = \lim \dfrac{\left(\dfrac{3}{4}\right)^n+5}{1+\left(\dfrac{2}{4}\right)^n} = \dfrac{5}{1} = 5.$

b). $\lim \dfrac{3^n-2.5^n}{7+3.5^n} = \lim \dfrac{\dfrac{3^n}{5^n}-\dfrac{2.5^n}{5^n}}{\dfrac{7}{5^n}+\dfrac{3.5^n}{5^n}} = \lim \dfrac{\left(\dfrac{3}{5}\right)^n-2}{\dfrac{7}{5^n}+3} = -\dfrac{2}{3}.$

c). $\lim \dfrac{2^n-3^n+5^{n+2}}{2^{n+1}+3^{n+2}+5^{n+1}} = \lim \dfrac{2^n-3^n+5^2.5^n}{2.2^n+3^2.3^n+5.5^n}$

$= \lim \dfrac{\dfrac{2^n}{5^n}-\dfrac{3^n}{5^n}+\dfrac{5^2.5^n}{5^n}}{\dfrac{2.2^n}{5^n}+\dfrac{3^2.3^n}{5^n}+\dfrac{5.5^n}{5^n}} = \lim \dfrac{\left(\dfrac{2}{5}\right)^n-\left(\dfrac{3}{5}\right)^n+25}{2.\left(\dfrac{2}{5}\right)^n+9.\left(\dfrac{3}{5}\right)^n+5} = 5.$

d). $\lim \dfrac{4.3^n+5^{n+1}}{3.2^n+5^n} = \lim \dfrac{4.3^n+5.5^n}{3.2^n+5^n} = \lim \dfrac{\dfrac{4.3^n}{5^n}+\dfrac{5.5^n}{5^n}}{\dfrac{3.2^n}{5^n}+\dfrac{5^n}{5^n}} = \lim \dfrac{4.\left(\dfrac{3}{5}\right)^n+5}{3.\left(\dfrac{2}{5}\right)^n+1} = 5.$

Câu 5: Tìm các giới hạn sau:

a). $\lim \dfrac{2^n+(-5)^n}{2.3^n+3.(-5)^n}$ b). $\lim \dfrac{\sqrt{9^n+1}}{3^n-1}$ c). $\lim \dfrac{(-1)^n.2^{5n+1}}{3^{5n+2}}$ d). $\lim \dfrac{n+\sqrt{n^2+1}}{n.3^n}$

LỜI GIẢI

a). $\lim \dfrac{2^n+(-5)^n}{2.3^n+3.(-5)^n}=\lim \dfrac{\dfrac{2^n}{(-5)^n}+\dfrac{(-5)^n}{(-5)^n}}{\dfrac{2.3^n}{(-5)^n}+\dfrac{3.(-5)^n}{(-5)^n}}=\lim \dfrac{\left(\dfrac{-2}{5}\right)^n+1}{2.\left(\dfrac{-3}{5}\right)^n+3}=\dfrac{1}{3}$.

b). $\lim \dfrac{\sqrt{9^n+1}}{3^n-1}=\lim \dfrac{\dfrac{\sqrt{9^n-1}}{3^n}}{\dfrac{3^n-1}{3^n}}=\lim \dfrac{\sqrt{1+\dfrac{1}{9^n}}}{1-\dfrac{1}{3^n}}=1.$

c). $\lim \dfrac{(-1)^n.2^{5n+1}}{3^{5n+2}}=\lim \dfrac{(-1)^n.2.2^{5n}}{3^2.3^{5n}}=\lim \dfrac{(-1).2}{9}.\left(\dfrac{2}{3}\right)^{5n}=0.$

d). $L=\lim \dfrac{n+\sqrt{n^2+1}}{n.3^n}=\lim \dfrac{\dfrac{n+\sqrt{n^2+1}}{n}}{\dfrac{n.3^n}{n}}=\lim \dfrac{1+\sqrt{1+\dfrac{1}{n^2}}}{3^n}=\lim \dfrac{1}{3^n}\left(1+\sqrt{1+\dfrac{1}{n^2}}\right)$. Có $\lim \dfrac{1}{n^2}=0$ nên

$\lim\left(1+\sqrt{1+\dfrac{1}{n^n}}\right)=2$ và $\lim \dfrac{1}{3^n}=0$. Do đó $L=0$.

Câu 6: Tìm các giới hạn sau:

a). $\lim \dfrac{n^2+4n-5}{3n^3+n^2+7}$ b). $\lim \dfrac{-2n^2+n+2}{3n^4+5}$ c). $\lim \dfrac{\sqrt{2n^2-n}}{1-3n^2}$ d). $\lim\left(\dfrac{\sin 3n}{4n}-1\right)$

LỜI GIẢI

a). $\lim \dfrac{n^2+4n-5}{3n^3+n^2+7}=\lim \dfrac{1+\dfrac{4}{n}-\dfrac{5}{n^2}}{3n+1+\dfrac{7}{n}}=\lim \dfrac{1}{3n+1}=0.$

b). $\lim \dfrac{-2n^2+n+2}{3n^4+5}=\lim \dfrac{-2+\dfrac{1}{n}+\dfrac{2}{n^2}}{3n^2+\dfrac{5}{n^2}}=\lim \dfrac{-2}{3n^2}=0.$

c). $\lim \dfrac{\sqrt{2n^2-n}}{1-3n^2}=\lim \dfrac{\dfrac{\sqrt{2n^2-n}}{n}}{\dfrac{1}{n}-3n}=\lim \dfrac{\sqrt{2-\dfrac{1}{n}}}{\dfrac{1}{n}-3n}=\lim \dfrac{\sqrt{2}}{-3n}=0.$

d). $\lim\left(\dfrac{\sin 3n}{4n}-1\right)=\lim \dfrac{\sin 3n}{4n}-1$

Ta có: $-1\le \sin 3n\le 1 \Leftrightarrow -\dfrac{1}{4n}\le \dfrac{\sin 3n}{4n}\le \dfrac{1}{4n}$

Mà: $\lim\left(-\dfrac{1}{4n}\right)=\lim \dfrac{1}{4n}=0\Rightarrow \lim \dfrac{\sin 3n}{4n}=0.$ Vậy $\lim\left(\dfrac{\sin 3n}{4n}-1\right)=-1.$

Câu 7: Tìm các giới hạn sau:

a). $\lim \dfrac{1}{\sqrt{3n+2}-\sqrt{2n+1}}$ b). $\lim \dfrac{5}{4^n+2^n}$ c). $\lim \dfrac{3^n+5.4^n}{7^n+2^n}$ d). $\lim \dfrac{(-5)^n+4^n}{(-7)^{n+1}+4^{n+1}}$

LỜI GIẢI

a). $\lim \dfrac{1}{\sqrt{3n+2}-\sqrt{2n+1}} = \lim \dfrac{1}{\sqrt{n\left(3+\dfrac{2}{n}\right)}-\sqrt{n\left(2+\dfrac{1}{n}\right)}}$

$= \lim \dfrac{1}{\sqrt{n}\left(\sqrt{3+\dfrac{2}{n}}-\sqrt{2+\dfrac{1}{n}}\right)} = \lim \dfrac{1}{\sqrt{n}\left(\sqrt{3}-\sqrt{2}\right)} = 0.$

b). $\lim \dfrac{5}{4^n+2^n} = \lim \dfrac{5.\dfrac{1}{4^n}}{1+\left(\dfrac{1}{2}\right)^n} = 0$. Do $\lim \dfrac{1}{4^n} = \lim \left(\dfrac{1}{4}\right)^n = 0$ và $\lim \left(\dfrac{1}{2}\right)^n = 0$.

c). $\lim \dfrac{3^n+5.4^n}{7^n+2^n} = \lim \dfrac{4^n\left(\dfrac{3^n}{4^n}+5\right)}{7^n\left(1+\dfrac{2^n}{7^n}\right)} = \lim \left(\dfrac{4}{7}\right)^n \left(\dfrac{\left(\dfrac{3}{4}\right)^n+5}{1+\left(\dfrac{2}{7}\right)^n}\right) = 0$. Do $\lim \left(\dfrac{3}{4}\right)^n = 0$, $\lim \left(\dfrac{2}{7}\right)^n = 0$ nên

$\lim \left(\dfrac{\left(\dfrac{3}{4}\right)^n+5}{1+\left(\dfrac{2}{7}\right)^n}\right) = 5$ và $\lim \left(\dfrac{4}{7}\right)^n = 0$. Nên $\lim u_n = 0$.

d). $\lim \dfrac{(-5)^n+4^n}{(-7)^{n+1}+4^{n+1}} = \lim \dfrac{(-5)^n\left(1+\dfrac{4^n}{(-5)^n}\right)}{(-7)^n\left(-7+\dfrac{4.4^n}{(-7)^n}\right)} = \lim \left(\dfrac{5}{7}\right)^n . \left(\dfrac{1+\left(-\dfrac{4}{5}\right)^n}{-7+4.\left(\dfrac{-4}{7}\right)^n}\right)$. Do $\lim \left(-\dfrac{4}{5}\right)^n = \lim \left(-\dfrac{4}{7}\right)^n = 0$

nên $\lim \left(\dfrac{1+\left(-\dfrac{4}{5}\right)^n}{-7+4.\left(\dfrac{-4}{7}\right)^n}\right) = -\dfrac{1}{7}$ và $\lim \left(\dfrac{5}{7}\right)^n = 0$.

Từ đó suy ra $\lim u_n = 0$.

Câu 8: Tìm các giới hạn sau:

a). $\lim\left(\sqrt{n^2-n}-n\right)$ b). $\lim\left(\sqrt{n^2+n+1}-n\right)$

c). $\lim\left(\sqrt{4n^2+n}-\sqrt{4n^2+2}\right)$ d). $\lim\left[n\left(\sqrt{n^2+1}-\sqrt{n^2+2}\right)\right]$

LỜI GIẢI

a). $\lim\left(\sqrt{n^2-n}-n\right) = \lim \dfrac{n^2-n-n^2}{\sqrt{n^2-n}+n}$

$= \lim \dfrac{-n}{\sqrt{n^2\left(1-\dfrac{1}{n}\right)}+n} = \lim \dfrac{-n}{n\left(\sqrt{1-\dfrac{1}{n}}+1\right)} = \lim \dfrac{-1}{\sqrt{1-\dfrac{1}{n}}+1} = \dfrac{1}{2}.$

b). $\lim\left(\sqrt{n^2+n+1}-n\right)=\lim\dfrac{n^2+n+1-n^2}{\sqrt{n^2+n+1}+n}$

$=\lim\dfrac{n+1}{\sqrt{n^2\left(1+\dfrac{1}{n}+\dfrac{1}{n^2}\right)}+n}=\lim\dfrac{n\left(1+\dfrac{1}{n}\right)}{n\left(\sqrt{1+\dfrac{1}{n}+\dfrac{1}{n^2}}+1\right)}=\lim\dfrac{1+\dfrac{1}{n}}{\sqrt{1+\dfrac{1}{n}+\dfrac{1}{n^2}}+1}=\dfrac{1}{2}.$

c). $\lim\left(\sqrt{4n^2+n}-\sqrt{4n^2+2}\right)=\lim\dfrac{4n^2-\left(4n^2+2\right)}{\sqrt{4n^2+n}+\sqrt{4n^2+2}}$

$=\lim\dfrac{n-2}{\sqrt{n^2\left(4+\dfrac{1}{n}\right)}+\sqrt{n^2\left(4+\dfrac{2}{n^2}\right)}}=\lim\dfrac{n-2}{n\sqrt{4+\dfrac{1}{n}}+n\sqrt{4+\dfrac{2}{n^2}}}$

$=\lim\dfrac{n\left(1-\dfrac{2}{n}\right)}{n\left(\sqrt{4+\dfrac{1}{n}}+\sqrt{4+\dfrac{2}{n^2}}\right)}=\lim\dfrac{1-\dfrac{2}{n}}{\sqrt{4+\dfrac{1}{n}}+\sqrt{4+\dfrac{2}{n^2}}}=\dfrac{1}{4}.$

d). $\lim\left[n\left(\sqrt{n^2+1}-\sqrt{n^2+2}\right)\right]=\lim\dfrac{n\left[\left(n^2+1\right)-\left(n+2\right)\right]}{\sqrt{n^2+1}+\sqrt{n^2+2}}$

$=\lim\dfrac{-n}{\sqrt{n^2\left(1+\dfrac{1}{n^2}\right)}+\sqrt{n^2\left(1+\dfrac{2}{n^2}\right)}}=\lim\dfrac{-n}{n\left(\sqrt{1+\dfrac{1}{n^2}}+\sqrt{1+\dfrac{2}{n^2}}\right)}$

$=\lim\dfrac{-1}{\sqrt{1+\dfrac{1}{n^2}}+\sqrt{1+\dfrac{2}{n^2}}}=\lim\dfrac{-1}{1+1}=-\dfrac{1}{2}.$

Câu 9: Tìm các giới hạn sau:

a). $\lim\left(\sqrt{n^2+2n}-n+3\right)$ b). $\lim\left(\sqrt{4n^2+3n+1}-2n+1\right)$

c). $\lim\left(1+n^2-\sqrt{n^4+3n+1}\right)$ d). $\lim\left[n\left(\sqrt{n+1}-\sqrt{n}\right)\right]$.

LỜI GIẢI

a). $\lim\left(\sqrt{n^2+2n}-n+3\right)=\lim\left(\sqrt{n^2+2n}-n\right)+3$

$=\lim\dfrac{n^2+2n-n^2}{\sqrt{n^2+2n}+n}+3=\lim\dfrac{2n}{\sqrt{n^2\left(1+\dfrac{2}{n}\right)}+n}+3$

$=\lim\dfrac{2n}{n\left(\sqrt{1+\dfrac{2}{n}}+1\right)}+3=\lim\dfrac{2}{\sqrt{1+\dfrac{2}{n}}+1}+3=\dfrac{2}{1+1}+3=4.$

b). $\lim\left(\sqrt{4n^2+3n+1}-2n+1\right)=\lim\left(\sqrt{4n^2+3n+1}-2n\right)+1$

$$=\lim\frac{4n^2+3n+1-4n^2}{\sqrt{4n^2+3n+1}+2n}+1=\lim\frac{3n+1}{\sqrt{n^2\left(4+\frac{3}{n}+\frac{1}{n^2}\right)}+2n}+1=\frac{3}{2+2}+1=\frac{7}{4}.$$

c). $\lim\left(1+n^2-\sqrt{n^4+3n+1}\right)=1+\lim\left(n^2-\sqrt{n^4+3n+1}\right)$

$$=1+\lim\frac{n^4-\left(n^4+3n+1\right)}{n^2+\sqrt{n^4+3n+1}}=1+\lim\frac{-3n-1}{n^2+\sqrt{n^4\left(1+\frac{3}{n^3}+\frac{1}{n^4}\right)}}$$

$$=1+\lim\frac{n\left(-3-\frac{1}{n}\right)}{n^2\left(1+\sqrt{1+\frac{3}{n^2}+\frac{1}{n^4}}\right)}=1+\lim\frac{-3}{n}=1+0=1.$$

d). $\lim\left[n\left(\sqrt{n+1}-\sqrt{n}\right)\right]=\lim\frac{n(n+1-n)}{\sqrt{n+1}+\sqrt{n}}$

$$=\lim\frac{n}{\sqrt{n\left(1+\frac{1}{n}\right)}+\sqrt{n}}=\lim\frac{n}{\sqrt{n}\left(\sqrt{1+\frac{1}{n}}+1\right)}=\lim\frac{\sqrt{n}}{2}=+\infty.$$

Câu 10: Tìm các giới hạn sau:

a). $\lim\left(\sqrt[3]{n+2}-\sqrt[3]{n}\right)$ b). $\lim\left(\sqrt[3]{n-n^3}+n+2\right)$

c). $\lim\left(\sqrt[3]{2n-n^3}+n-1\right)$ d). $\lim\left(\sqrt[3]{n^3-2n^2}-n-1\right)$

LỜI GIẢI

a). $\lim\left(\sqrt[3]{n+2}-\sqrt[3]{n}\right)=\lim\frac{n+2-n}{\left(\sqrt[3]{n+2}\right)^2+\sqrt[3]{n+2}.\sqrt[3]{n}+\left(\sqrt[3]{n}\right)^2}$

$$=\lim\frac{2}{\left(\sqrt[3]{n\left(1+\frac{2}{n}\right)}\right)^2+\sqrt[3]{n\left(1+\frac{2}{n}\right)}.\sqrt[3]{n}+\left(\sqrt[3]{n}\right)^2}$$

$$=\lim\frac{2}{\left(\sqrt[3]{n}\right)^2\left[\left(\sqrt[3]{1+\frac{2}{n}}\right)^2+\sqrt[3]{1+\frac{2}{n}}+1\right]}=\lim\frac{2}{3\left(\sqrt[3]{n}\right)^2}=0.$$

b). $\lim\left(\sqrt[3]{n-n^3}+n+2\right)=\lim\left(\sqrt[3]{n-n^3}+n\right)+2$

$$=\lim\frac{n-n^3+n^3}{\left(\sqrt[3]{n-n^3}\right)^2-\sqrt[3]{n-n^3}.n+n^2}+2=\lim\frac{n}{\left(\sqrt[3]{n^3\left(\frac{1}{n^2}-1\right)}\right)^2-\sqrt[3]{n^3\left(\frac{1}{n^2}-1\right)}.n+n^2}+2$$

$$=\lim\frac{n}{n^2\left[\left(\sqrt[3]{\frac{2}{n^2}-1}\right)^2-\sqrt[3]{\frac{1}{n^2}-1}+1\right]}+2=\lim\frac{1}{3n}+2=0+2=2.$$

c). $\lim\left(\sqrt[3]{2n-n^3}+n-1\right)=\lim\left(\sqrt[3]{2n-n^3}+n\right)-1$

$=\lim\dfrac{2n-n^3+n^3}{\left(\sqrt[3]{2n-n^3}\right)^2-\sqrt[3]{2n-n^3}.n+n^2}-1\ =\lim\dfrac{2n}{\left(\sqrt[3]{n^3\left(\frac{2}{n^2}-1\right)}\right)^3-\sqrt[3]{n^3\left(\frac{2}{n^2}-1\right)}.n+n^2}-1$

$=\lim\dfrac{2n}{n^2\left[\left(\sqrt[3]{\frac{2}{n^2}-1}\right)^2-\sqrt[3]{\frac{2}{n^2}-1}+1\right]}-1\ =\lim\dfrac{2}{3n}-1=0-1=-1.$

d) $\lim\left(\sqrt[3]{n^3-2n^2}-n-1\right)=\lim\left(\sqrt[3]{n^3-2n^2}-n\right)-1$

$=\lim\dfrac{n^3-2n^2-n^3}{\left(\sqrt[3]{n^3-2n}\right)^2+\sqrt[3]{n^3-2n}.n+n^2}-1\ =\lim\dfrac{-2n^2}{\left(\sqrt[3]{n^3\left(1-\frac{2}{n^2}\right)}\right)^2+\sqrt{n^3\left(1-\frac{2}{n^2}\right)}.n+n^2}-1$

$=\lim\dfrac{-2n^2}{n^2\left[\left(\sqrt[3]{1-\frac{2}{n^2}}\right)^2+\sqrt[3]{1-\frac{2}{n^2}}+1\right]}-1\ =\lim\dfrac{-2}{\left(\sqrt[3]{1-\frac{2}{n^2}}\right)^2+\sqrt[3]{1-\frac{2}{n^2}}+13}-1=-\dfrac{2}{3}+1=\dfrac{1}{3}.$

Câu 11: Tìm các giới hạn sau:

a). $\lim\left(\sqrt[3]{8n^3+3n^2-2}+5-2n\right)$ b) $\lim\left(\sqrt[3]{8n^3+3n^2-2}+\sqrt[3]{5n^2-8n^3}\right)$ c) $\lim\left[n.\left(\sqrt[3]{n^3+n}-n\right)\right]$

d). $\lim\left(\sqrt[3]{8n^3+2n^2-1}+3-2n\right)$

LỜI GIẢI

a). $\lim\left(\sqrt[3]{8n^3+3n^2-2}+5-2n\right)=\lim\left(\sqrt[3]{8n^3+3n^2-2}-2n\right)+5$

$=\lim\dfrac{8n^3+3n^2-2-8n^3}{\left(\sqrt[3]{8n^3+3n^2-2}\right)^2+\sqrt[3]{8n^3+3n^2-2}.2n+4n^2}+5$

$=\lim\dfrac{3n^2-2}{\left(\sqrt[3]{n^3\left(8+\frac{3}{n}-\frac{2}{n^3}\right)}\right)^2+\sqrt{n^3\left(8+\frac{3}{n}-\frac{2}{n^3}\right)}.2n+4n^2}+5$

$=\lim\dfrac{n^2\left(3-\frac{2}{n^2}\right)}{n^2\left[\left(\sqrt[3]{8+\frac{3}{n}-\frac{2}{n^3}}\right)^2+\sqrt[3]{8+\frac{3}{n}-\frac{2}{n^3}}.2+4\right]}+5$

$=\lim\dfrac{3-\frac{2}{n^2}}{\left(\sqrt[3]{8+\frac{3}{n}-\frac{2}{n^2}}\right)+\sqrt[3]{8+\frac{3}{n}-\frac{2}{n^3}}.2+4}+5\ =\dfrac{3}{4+4+4}+5=\dfrac{1}{4}+5=\dfrac{21}{4}.$

b). $\lim\left(\sqrt[3]{8n^3+3n^2-2}+\sqrt[3]{5n^2-8n^3}\right)$

$$= \lim \frac{8n^3+3n^2-2+5n^2-8n^3}{\left(\sqrt[3]{8n^3+3n^2-2}\right)^2-\sqrt[3]{8n^3+3n^2-2}.\sqrt[3]{5n^2-8n^3}+\left(\sqrt[3]{5n^2-8n^3}\right)^2}$$

$$= \lim \frac{8n^2-2}{\left(\sqrt[3]{n^3\left(8+\frac{3}{n}-\frac{2}{n^3}\right)}\right)^2-\sqrt[3]{n^3\left(8+\frac{3}{n}-\frac{2}{n^3}\right)}\sqrt[5]{n^3\left(\frac{5}{n}-8\right)}+\left(\sqrt[3]{n^3\left(\frac{5}{n}-8\right)}\right)^2}$$

$$= \lim \frac{n^2\left(8-\frac{2}{n^2}\right)}{n^2\left[\left(\sqrt[3]{8+\frac{3}{n}-\frac{2}{n^2}}\right)^2-\sqrt[3]{8+\frac{3}{n}-\frac{2}{n^3}}.\sqrt[3]{\frac{5}{n}-8}+\left(\sqrt[3]{\frac{5}{n}-8}\right)^3\right]}$$

$$= \lim \frac{8-\frac{2}{n^2}}{\left(\sqrt[3]{8+\frac{3}{n}-\frac{2}{n^2}}\right)^2-\sqrt[3]{8+\frac{3}{n}-\frac{2}{n^3}}.\sqrt[3]{\frac{5}{n}-8}+\left(\sqrt[3]{\frac{5}{n}-8}\right)^3}=\frac{8}{4+4+4}=\frac{2}{3}.$$

c) $\lim\left[n.\left(\sqrt[3]{n^3+n}-n\right)\right]=\lim\dfrac{n\left(n^3+n-n^3\right)}{\left(\sqrt[3]{n^3+n}\right)^2+\sqrt[3]{n^3+n}.n+n^2}$

$$= \lim \frac{n^2}{\left(\sqrt[3]{n^3\left(1+\frac{2}{n^2}\right)}\right)^2+\sqrt[3]{n^3\left(1+\frac{1}{n^2}\right)}.n+n^2}$$

$$= \lim \frac{n^2}{n^2\left[\left(\sqrt[3]{1+\frac{2}{n^2}}\right)^2+\sqrt[3]{1+\frac{1}{n^2}}+1\right]}=\lim\frac{1}{\left(\sqrt[3]{1+\frac{2}{n^2}}\right)^2+\sqrt[3]{1+\frac{1}{n^2}}+1}=\frac{1}{3}.$$

d). Hoàn toàn tương tự câu a).

Câu 12: Tìm các giới hạn sau:

a). $\lim\dfrac{1}{\sqrt{n+2}-\sqrt{n+1}}$ b). $\lim\dfrac{1}{\sqrt{3n^2+2n}-\sqrt{3n^2+1}}$

c). $\lim\left(n+\sqrt[3]{1-n^3}\right)$ d) $\lim\left(\sqrt[3]{8n^3+3n^2+4}-2n+1\right)$

LỜI GIẢI

a). $\lim\dfrac{1}{\sqrt{n+2}-\sqrt{n+1}}=\lim\dfrac{\sqrt{n+2}+\sqrt{n+1}}{n+2-(n+1)}$

$$= \lim\left(\sqrt{n\left(1+\frac{2}{n}\right)}+\sqrt{n\left(1+\frac{1}{n}\right)}\right)=\lim\sqrt{n}\left(\sqrt{1+\frac{1}{n}}+\sqrt{1+\frac{1}{n}}\right)$$

$$= \lim\left(2\sqrt{n}\right)=+\infty.$$

b). $\lim\dfrac{1}{\sqrt{3n^2+2n}-\sqrt{3n^2+1}}=\lim\dfrac{\sqrt{3n^2+2n}+\sqrt{3n^2+1}}{\left(3n^2+2\right)-\left(3n^2+1\right)}$

$$=\lim\frac{\sqrt{n^2\left(3+\frac{2}{n}\right)}+\sqrt{n^2\left(3+\frac{1}{n^2}\right)}}{2n-1}=\lim\frac{n\left(\sqrt{3+\frac{2}{n}}+\sqrt{3+\frac{1}{n^2}}\right)}{n\left(2-\frac{1}{n}\right)}$$

$$=\lim\frac{\sqrt{3+\frac{2}{n}}+\sqrt{3+\frac{1}{n^2}}}{2-\frac{2}{n}}=\frac{\sqrt{3}+\sqrt{3}}{2}=\frac{2\sqrt{3}}{2}=\sqrt{3}.$$

c). $\lim\left(n+\sqrt[3]{1-n^3}\right)=\lim\dfrac{n^3+1-n^3}{n^2-n\sqrt[3]{1-n^3}+\left(\sqrt[3]{1-n^3}\right)^2}$

$$=\lim\frac{1}{n^2-n\sqrt[3]{n^3\left(\frac{1}{n^3}-1\right)}+\left(\sqrt[3]{n^3\left(\frac{1}{n^3}-1\right)}\right)^2}$$

$$=\lim\frac{1}{n^2\left[1-\sqrt[3]{\frac{1}{n^3}-1}+\left(\sqrt[3]{\frac{1}{n^3}-1}\right)^2\right]}=\lim\frac{1}{3n^2}=0.$$

d). $\lim\left(\sqrt[3]{8n^3+3n^2+4}-2n+1\right)=\lim\left(\sqrt[3]{8n^3+3n^2+4}-2n\right)+1$

$$=\lim\frac{8n^3+3n^2+4-8n^3}{\left(\sqrt[3]{8n^3+3n^2+4}\right)^2+\sqrt[3]{8n^3+3n^2+4}.2n+4n^2}+1$$

$$=\lim\frac{3n^2+4}{n^2\left(\sqrt[3]{8+\frac{3}{n}+\frac{4}{n^3}}\right)^2+2n^2.\sqrt[3]{8+\frac{3}{n}+\frac{4}{n^3}}+4n^2}+1$$

$$=\lim\frac{n^2\left(3+\frac{4}{n^2}\right)}{n^2\left(\sqrt[3]{8+\frac{3}{n}+\frac{4}{n^3}}\right)^2+2n^2.\sqrt[3]{8+\frac{3}{n}+\frac{4}{n^3}}+4n^2}+1$$

$$=\lim\frac{3+\frac{4}{n^2}}{\left(\sqrt[3]{8+\frac{3}{n}+\frac{4}{n^3}}\right)^2+2.\sqrt[3]{8+\frac{3}{n}+\frac{4}{n^3}}+4}+1=\frac{1}{4}+1=\frac{5}{4}.$$

Câu 13*: Tìm các giới hạn sau:

a) $\lim\dfrac{\sqrt{4n^2+1}-2n}{\sqrt{n^2+4n+1}-n}$

b) $\lim\dfrac{\sqrt{4n^2+1}-2n}{\sqrt[3]{n^3+4n+1}-n}$

c) $\lim\dfrac{n\left(\sqrt[3]{4-n^3}+n\right)}{\sqrt{4n^2+1}-2n}$

d) $\lim\dfrac{n^2+\sqrt[3]{1-n^6}}{\sqrt{n^4+1}-n^2}$

LỜI GIẢI

a) $\lim \dfrac{\sqrt{4n^2+1}-2n}{\sqrt{n^2+4n+1}-n}$

Ta có: $\sqrt{4n^2+1}-2n=\dfrac{4n^2+1-4n^2}{\sqrt{4n^2+1}+2n}=\dfrac{1}{\sqrt{4n^2+1}+2n}$

Ta có: $\dfrac{1}{\sqrt{n^2+4n+1}-n}=\dfrac{\sqrt{n^2+4n+1}+n}{n^2+4n+1-n^2}=\dfrac{\sqrt{n^2+4n+1}+n}{4n+1}$

Vậy $\lim \dfrac{\sqrt{n^2+4n+1}+n}{\left(\sqrt{4n^2+1}+2n\right)(2n+1)}=\lim \dfrac{\sqrt{n^2\left(1+\frac{4}{n}+\frac{1}{n^2}\right)}+n}{\left(\sqrt{n^2\left(4+\frac{1}{n^2}\right)}+2n\right)(2n+1)}$

$=\lim \dfrac{n\left[\sqrt{1+\frac{4}{n}+\frac{1}{n^2}}+1\right]}{n\left(\sqrt{4+\frac{1}{n^2}}+2\right)n\left(2+\frac{1}{n}\right)}=\lim \dfrac{\sqrt{1+\frac{4}{n}+\frac{1}{n^2}}+1}{n\left(\sqrt{4+\frac{1}{n^2}}+2\right)\left(2+\frac{1}{n}\right)}$

$=\lim \dfrac{2}{n(2+2)2}=\lim \dfrac{1}{4n}=0.$

b). $\lim \dfrac{\sqrt{4n^2+1}-2n}{\sqrt[3]{n^3+4n+1}-n}=\lim \dfrac{\left(4n^2+1-4n^2\right)\left[\left(\sqrt[3]{n^3+4n+1}\right)^2+n\sqrt[3]{n^3+4n+1}+n^2\right]}{\left(\sqrt{4n^2+1}+2n\right)\left(n^3+4n+1-n^3\right)}$

$=\lim \dfrac{\left(\sqrt[3]{n^3\left(1+\frac{4}{n^2}+\frac{1}{n^3}\right)}\right)^2+n\sqrt[3]{n^3\left(1+\frac{4}{n^2}+\frac{1}{n^3}\right)}+n^2}{\left(\sqrt{n^2\left(4+\frac{1}{n^2}\right)}+2n\right)(4n+1)}$

$=\lim \dfrac{n^2\left[\left(\sqrt[3]{1+\frac{4}{n^2}+\frac{1}{n^3}}\right)^2+\sqrt[3]{1+\frac{4}{n^2}+\frac{1}{n^3}}+1\right]}{n\left(\sqrt{4+\frac{1}{n^2}}+2\right)n\left(4+\frac{1}{n}\right)}$

$=\lim \dfrac{\left(\sqrt[3]{1+\frac{4}{n^2}+\frac{1}{n^3}}\right)^2+\sqrt[3]{1+\frac{4}{n^2}+\frac{1}{n^3}}+1}{\left(\sqrt{4+\frac{1}{n^2}}+2\right)\left(4+\frac{1}{n}\right)}=\lim \dfrac{1+1+1}{(2+2)4}=\dfrac{3}{16}.$

c). $\lim \dfrac{n\left(\sqrt[3]{4-n^3}+n\right)}{\sqrt{4n^2+1}-2n}=\lim \dfrac{n\left(4-n^3+n^3\right)}{\left(\sqrt[3]{4-n^3}\right)^2-n\sqrt[3]{4-n^3}+n^2}.\dfrac{\sqrt{4n^2+1}+2n}{4n^2+1-4n^2}$

$$=\lim\frac{4n\left[\sqrt{n^2\left(1+\frac{1}{n^2}\right)}+2n\right]}{\left(\sqrt[3]{n^3\left(\frac{4}{n^3}-1\right)}\right)^2-n\sqrt[3]{n^3\left(\frac{4}{n^3}-1\right)}+n^2}=\lim\frac{4n^2\left(\sqrt{4+\frac{1}{n^2}}+2\right)}{n^2\left[\left(\sqrt[3]{\frac{4}{n^3}-1}\right)^2-\sqrt[3]{\frac{4}{n^3}-1}+1\right]}$$

$$=\lim\frac{4\left(\sqrt{4+\frac{1}{n^2}}+2\right)}{\left(\sqrt[3]{\frac{4}{n^3}-1}\right)^2-\sqrt[3]{\frac{4}{n^3}-1}+1}=\frac{4(2+2)}{1+1+1}=\frac{16}{3}.$$

d). $\lim\frac{n^2+\sqrt[3]{1-n^6}}{\sqrt{n^4+1}-n^2}=\lim\frac{n^6+1-n^6}{n^4-n^2.\sqrt[3]{1-n^6}}.\frac{\sqrt{n^4+1}+n^2}{n^4+1-n^4}$

$$=\lim\frac{\sqrt{n^4+1}+n^2}{n^4-n^2\sqrt[3]{1-n^6}+\left(\sqrt[3]{1-n^6}\right)^2}=\lim\frac{\sqrt{n^4\left(1+\frac{1}{n^4}\right)}+n^2}{n^4-n^2\sqrt[3]{1-n^6}+\left(\sqrt[3]{1-n^6}\right)^2}$$

$$=\lim\frac{n^2\left(\sqrt{1+\frac{1}{n^4}}+1\right)}{n^4\left[1-\sqrt[3]{\frac{1}{n^6}-1}+\left(\sqrt[3]{\frac{1}{n^6}-1}\right)^2\right]}=\lim\frac{2}{3n^2}=0.$$

Câu 14: Tìm các giới hạn sau:

a). $\lim\frac{1+2+3+...+n}{3n^2+1}$ b). $\lim\frac{1+3+5+7+...+(2n-1)}{n^2+3n+1}$

c). $\lim\frac{2+5+8+...+(3n-1)}{4n^2+1}$ d). $\lim\frac{1+2+2^2+2^3+...+2^n}{1+3+3^2+3^3+...+3^n}$

e). $\lim\left[\frac{1}{1\sqrt{2}+2\sqrt{1}}+\frac{1}{2\sqrt{3}+3\sqrt{2}}+\cdots+\frac{1}{n\sqrt{n+1}+(n+1)\sqrt{n}}\right]$

LỜI GIẢI

a). $\lim\frac{1+2+3+...+n}{3n^2+1}$ (1). Ta có: $1+2+3+...+n=\frac{n(n+1)}{2}$ (đã chứng minh bằng phương pháp quy nạp ở chương III).

Vậy $(1)\Leftrightarrow\lim\frac{\frac{n(n+1)}{2}}{3n^2+1}=\lim\frac{n^2\left(1+\frac{1}{n}\right)}{n^2\left(3+\frac{1}{n^2}\right)}=\lim\frac{1+\frac{1}{n^2}}{2\left(3+\frac{1}{n^2}\right)}=\frac{1}{6}.$

b). $\lim\frac{1+3+5+7+...+(2n-1)}{n^2+3n+1}$ (1)

Ta có: $1+3+5+7+...(2n-1)=n^2$ (đã chứng minh bằng phương pháp quy nạp ở chương III).

Vậy $(1)\Leftrightarrow\lim\frac{n^2}{n^2+3n+1}=\lim\frac{1}{1+\frac{3}{n}+\frac{1}{n^2}}=1.$

c) $\lim \dfrac{2+5+8+...+(3n-1)}{4n^2+1}$ (1)

Ta có dãy số: $2;5;8;...;(3n-1)$ là một cấp số cộng với $u_1=2, u_2=5 \Rightarrow d=3$. Số hạng tổng quát:

$u_m = u_1+(m-1)d \Leftrightarrow 3n-1=2+(m-1).3$

$\Leftrightarrow 3n-1=3m-1 \Leftrightarrow n=m \Rightarrow$ cấp số cộng có n số hạng.

$$S_n=\frac{n\left[2u_1+(n-1)d\right]}{2}=\frac{n\left[4+(n-1)3\right]}{2}=\frac{n(3n+1)}{2}$$

$$(1)\Leftrightarrow \lim\frac{\frac{n(3n+1)}{2}}{4n^2+1}=\lim\frac{3+\frac{1}{n}}{2\left(4+\frac{1}{n^2}\right)}=\frac{3}{8}.$$

d) $\lim \dfrac{1+2+2^2+2^3+...+2^n}{1+3+3^2+3^3+...+3^n}$ (1)

Ta có: $1,2,2^2,2^3,...,2^n$ là một dãy số thuộc cấp số nhân, với $u_1=1, q=2$.

Số hạng tổng quát: $u_m=2^n=u_1.q^{m-1} \Leftrightarrow 2^n=2^{m-1} \Leftrightarrow n=m-1 \Rightarrow m=n+1$.

Vậy cấp số nhân này có $(n+1)$ số hạng

$$S_m=u_1.\frac{1-q^m}{1-q}=\frac{1-2^{n+1}}{1-2}=2^{n+1}-1$$

Tương tự ta tính được: $1+3+3^2+3^3+...+3^n=\dfrac{3^{n+1}-1}{2}$.

$$(1)\Leftrightarrow \lim\frac{2^{n+1}-1}{\frac{3^{n+1}-1}{2}}=\lim 2\frac{\left(\frac{2}{3}\right)^{n+1}-3^{\frac{1}{n+1}}}{1-\frac{1}{3^{n+1}}}=2.0=0.$$

e). $L=\lim\left[\dfrac{1}{1\sqrt{2}+2\sqrt{1}}+\dfrac{1}{2\sqrt{3}+3\sqrt{2}}+\cdots+\dfrac{1}{n\sqrt{n+1}+(n+1)\sqrt{n}}\right]$

Ta có: $\dfrac{1}{n\sqrt{n+1}+(n+1)\sqrt{n}}=\dfrac{1}{\sqrt{n}\sqrt{n+1}\left(\sqrt{n}+\sqrt{n+1}\right)}=\dfrac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n}\sqrt{n+1}}=\dfrac{1}{\sqrt{n}}-\dfrac{1}{\sqrt{n+1}}$

Vậy: $\dfrac{1}{1\sqrt{2}+2\sqrt{1}}+\dfrac{1}{2\sqrt{3}+3\sqrt{2}}+\cdots+\dfrac{1}{n\sqrt{n+1}+(n+1)\sqrt{n}}$

$=\dfrac{1}{\sqrt{1}}-\dfrac{1}{\sqrt{2}}+\dfrac{1}{\sqrt{2}}-\dfrac{1}{\sqrt{3}}+\cdots+\dfrac{1}{\sqrt{n}}-\dfrac{1}{\sqrt{n+1}}=1-\dfrac{1}{\sqrt{n+1}}$

Vậy $L=\lim\left(1-\dfrac{1}{\sqrt{n+1}}\right)=1$

Câu 15: Tìm các giới hạn sau:

a). $\lim \dfrac{1+\frac{1}{2}+\frac{1}{4}+...+\frac{1}{2^n}}{1+\frac{1}{3}+\frac{1}{9}+...+\frac{1}{3^n}}$ b) $\lim \dfrac{\sqrt{1+4+7+...+(3n+1)}}{2n^2+\sqrt{n^4+2n+1}}$ c) $\lim\left[\dfrac{1}{1.3}+\dfrac{1}{3.5}+...+\dfrac{1}{(2n-1)(2n+1)}\right]$ d) $\lim\left[\dfrac{1}{1.3}+\dfrac{1}{2.4}+...+\dfrac{1}{n(n+2)}\right]$

LỜI GIẢI

a). $\lim \dfrac{1+\frac{1}{2}+\frac{1}{4}+...+\frac{1}{2^n}}{1+\frac{1}{3}+\frac{1}{9}+...+\frac{1}{3^n}}$ (1)

Ta tính tổng: $1+\frac{1}{2}+\frac{1}{4}+...+\frac{1}{2^n}$. Ta có: $u_1=1, u_2=\frac{1}{2}\Rightarrow q=\frac{u_2}{u_1}=\frac{1}{2}$.

Số hạng tổng quát: $u_m=\frac{1}{2^n}=u_1.q^{m-1}$

$\Leftrightarrow \frac{1}{2^n}=\left(\frac{1}{2}\right)^{m-1} \Leftrightarrow \left(\frac{1}{2}\right)^n=\left(\frac{1}{2}\right)^{m-1} \Leftrightarrow n=m-1 \Leftrightarrow m=n+1.$

$$S_m=u_1.\frac{q^m-1}{q-1}=\frac{\left(\frac{1}{2}\right)^{m+1}-1}{\frac{1}{2}-1}=\left(1-\left(\frac{1}{2}\right)^{n+1}\right).2$$

Tương tự tổng: $1+\frac{1}{3}+\frac{1}{5}+...+\frac{1}{3^n}=\frac{3}{2}\left(1-\left(\frac{1}{3}\right)^{n+1}\right)$

Vậy $(1)\Leftrightarrow \lim\dfrac{\left(1-\frac{1}{2^{n+1}}\right).2}{\left(1-\frac{1}{3^{n+1}}\right).\frac{3}{2}}=\dfrac{2}{\frac{3}{2}}=\dfrac{4}{3}.$

b) $\lim \dfrac{\sqrt{1+4+7+...+(3n+1)}}{2n^2+\sqrt{n^4+2n+1}}$

Ta tính tổng: $1+4+7+...(3n+1)$. Ta có: $u_1=1, u_2=4\Rightarrow d=u_2-u_1=3$

Số hạng tổng quát: $u_m=3n+1=u_1+(m-1)d$

$\Leftrightarrow 3n+1=1+(m-1)3 \Leftrightarrow 3n+1=3m-2\Rightarrow m=n+1$

$$S_m=\frac{m}{2}\left[2m+(m-1)d\right]=\frac{n+1}{2}(2+3n)=\frac{(n+1)(3n+2)}{2}$$

Vậy: $\lim\dfrac{\frac{(n+1)(3n+2)}{2}}{2n^2+\sqrt{n^4+2n+1}}=\lim\dfrac{n^2\left(1+\frac{1}{n}\right)\left(3+\frac{2}{n}\right)}{n^2\left(4+2\sqrt{1+\frac{2}{n^3}+\frac{1}{n^4}}\right)}$

$=\lim\dfrac{\left(1+\dfrac{1}{n}\right)\left(3+\dfrac{2}{n}\right)}{4+2\sqrt{1+\dfrac{2}{n^3}+\dfrac{1}{n^4}}}=\dfrac{3}{6}=\dfrac{1}{2}.$

c). $\lim\left[\dfrac{1}{1.3}+\dfrac{1}{3.5}+...+\dfrac{1}{(2n-1)(2n+1)}\right]$

$=\lim\dfrac{1}{2}\left[\dfrac{2}{1.3}+\dfrac{2}{3.5}+...+\dfrac{2}{(2n-1)(2n+1)}\right]$

$=\lim\dfrac{1}{2}\left[\dfrac{3-1}{1.3}+\dfrac{5-3}{3.5}+...+\dfrac{(2n+1)-(2n-1)}{(2n-1)(2n+1)}\right]$

$=\lim\dfrac{1}{2}\left(1-\dfrac{1}{3}+\dfrac{1}{3}-\dfrac{1}{5}+...+\dfrac{1}{2n-1}-\dfrac{1}{2n+1}\right)$

$=\lim\dfrac{1}{2}\left(1-\dfrac{1}{2n+1}\right)=\dfrac{1}{2}.$

d). $\lim\left[\dfrac{1}{1.3}+\dfrac{1}{2.4}+...+\dfrac{1}{n(n+2)}\right]$

$=\lim\dfrac{1}{2}\left[\dfrac{2}{1.3}+\dfrac{2}{2.4}+...+\dfrac{2}{n(n+2)}\right]$

$=\lim\dfrac{1}{2}\left[\dfrac{3-1}{1.3}+\dfrac{4-2}{2.4}+...+\dfrac{n+2-n}{n(n+2)}\right]$

$=\lim\dfrac{1}{2}\left(1-\dfrac{1}{3}+\dfrac{1}{2}-\dfrac{1}{4}+...+\dfrac{1}{n}-\dfrac{1}{n+2}\right)=\lim\dfrac{1}{2}\left(1-\dfrac{1}{n+2}\right)=\dfrac{1}{2}.$

Câu 16: Tìm các giới hạn sau:

a). $\lim\left(3n-5-\sqrt{9n^2+1}\right)$ b). $\lim\left(\sqrt{n^2+n+1}-\sqrt[3]{n^3+n^2}\right)$ c). $\lim\left(\sqrt[3]{8n^3+n^2}-\sqrt{4n^2-n+5}\right)$ d). $\lim\dfrac{(n+1)^4-(n-1)^4}{(n+1)^4+(n-1)^4}$

e). $\lim\dfrac{(2n+1)^4-(n-1)^4}{(2n+1)^4+(n-1)^4}$ f). $\lim\dfrac{\left(\sqrt{n^2+1}+n\right)^2}{\sqrt[3]{n^6+1}}$

LỜI GIẢI

a). $\lim\left(3n-5-\sqrt{9n^2+1}\right)=\lim\left(3n-\sqrt{9n^2+1}\right)-5$

$=\lim\dfrac{9n^2-(9n^2+1)}{3n+\sqrt{9n^2+1}}-5=\lim\dfrac{-1}{3n+n\sqrt{9+\dfrac{1}{n^2}}}-5$

$=\lim\dfrac{-1}{6n}-5=0-5=-5.$

b). $\lim\left(\sqrt{n^2+n+1}-\sqrt[3]{n^3+n^2}\right)$

$$=\lim\left(\sqrt{n^2+n+1}-n+n-\sqrt[3]{n^3+n^2}\right)$$

$$=\lim\left(\sqrt{n^2+n+1}-n\right)+\lim\left(n-\sqrt[3]{n^3+n^2}\right)$$

$$=\lim\frac{n^2+n+1-n^2}{\sqrt{n^2+n+1}+n}+\lim\frac{n^3-\left(n^3+n^2\right)}{n^2+n\sqrt[3]{n^3+n^2}+\left(\sqrt[3]{n^3+n^2}\right)^2}$$

$$=\lim\frac{n+1}{\sqrt{n^2\left(1+\frac{1}{n}+\frac{1}{n^2}\right)}+n}+\lim\frac{-n^2}{n^2+n\sqrt[3]{n^3\left(1+\frac{1}{n}\right)}+\left(\sqrt[3]{n^3\left(1+\frac{1}{n}\right)}\right)^2}$$

$$=\lim\frac{n\left(1+\frac{1}{n}\right)}{n^2\left(\sqrt{1+\frac{1}{n}+\frac{1}{n^2}}+1\right)}+\lim\frac{-n^2}{n^2\left[1+\sqrt[3]{1+\frac{1}{n}}+\left(\sqrt[3]{1+\frac{1}{n}}\right)^2\right]}$$

$$=\lim\frac{1+\frac{1}{n}}{\sqrt{1+\frac{1}{n}+\frac{1}{n^2}}+1}+\lim\frac{-1}{1+\sqrt[3]{1+\frac{1}{n}}+\left(\sqrt[3]{1+\frac{1}{n}}\right)^2}=\frac{1}{2}-\frac{1}{3}=\frac{1}{6}.$$

c). $\lim\left(\sqrt[3]{8n^3+n^2}-\sqrt{4n^2-n+5}\right)$

$$=\lim\left(\sqrt[3]{8n^3+n^2}-2n+2n-\sqrt{4n^2-n+5}\right)$$

$$=\lim\left(\sqrt[3]{8n^3+n^2}-2n\right)+\lim\left(2n-\sqrt{4n^2-n+5}\right)$$

- Tính $\lim\left(\sqrt[3]{8n^3+n^2}-2n\right)=\lim\dfrac{8n^3+n^2-8n^3}{\left(\sqrt[3]{8n^3+n^2}\right)^2+\sqrt[3]{8n^3+n^2}.2n+4n^2}$

$$=\lim\frac{n^2}{\left(\sqrt[3]{n^3\left(8+\frac{1}{n}\right)}\right)^2+\sqrt[3]{n^3\left(8+\frac{1}{n}\right)}.2n+4n^2}$$

$$=\lim\frac{n^2}{n^2\left[\left(\sqrt[3]{8+\frac{1}{n}}\right)^2+\sqrt[3]{8+\frac{1}{n}}.2+4\right]}=\lim\frac{1}{\left(\sqrt[3]{8+\frac{1}{n}}\right)^2+\sqrt[3]{8+\frac{1}{n}}.2+4}=\frac{1}{12}.$$

- Tính $\lim\left(2n-\sqrt{4n^2-n+5}\right)$

$$=\lim\frac{4n^2-\left(4n^2-n+5\right)}{2n+\sqrt{4n^2-n+5}}=\lim\frac{n-5}{2n+\sqrt{n^2\left(4-\frac{1}{n}+\frac{5}{n^2}\right)}}$$

$$=\lim\frac{n\left(1-\frac{5}{n}\right)}{n\left(2+\sqrt{4-\frac{1}{n}+\frac{5}{n^2}}\right)}=\lim\frac{1-\frac{5}{n}}{2+\sqrt{4-\frac{1}{n}+\frac{5}{n^2}}}=\frac{1}{4}.$$

Vậy giới hạn cần tìm là: $\frac{1}{12}+\frac{1}{4}=\frac{4}{12}=\frac{1}{3}$.

d). $$\lim\frac{(n+1)^4-(n-1)^4}{(n+1)^4+(n-1)^4}=\lim\frac{\left[(n+1)^2-(n-1)^2\right].\left[(n+1)^2+(n-1)^2\right]}{(n+1)^4+(n-1)^4}$$

$$=\lim\frac{\left[(n+1)-(n-1)\right]\left[(n+1)+(n-1)\right]\left[(n+1)^2+(n-1)^2\right]}{(n+1)^4+(n-1)^4}$$

$$=\lim\frac{4n\left[n^2\left(1+\frac{1}{n}\right)^2+n^2\left(1-\frac{1}{n}\right)^2\right]}{n^4\left(1+\frac{1}{n}\right)^4+n^4\left(1-\frac{1}{n}\right)^4}=\lim\frac{4n^3\left[\left(1+\frac{1}{n}\right)^2+\left(1-\frac{1}{n}\right)^2\right]}{n^4\left[\left(1+\frac{1}{n^4}\right)+\left(1-\frac{1}{n}\right)^4\right]}=\lim\frac{8}{2n}=0.$$

e). $$\lim\frac{(2n+1)^4-(n-1)^4}{(2n+1)^4+(n-1)^4}=\lim\frac{n^4\left(2+\frac{1}{n}\right)-n^4\left(1-\frac{1}{n}\right)^4}{n^4\left(2+\frac{1}{n}\right)^4+n^4\left(1-\frac{1}{n}\right)^4}$$

$$=\lim\frac{\left(2+\frac{1}{n}\right)^4-\left(1-\frac{1}{n}\right)^4}{\left(2+\frac{1}{n}\right)^4+\left(1-\frac{1}{n}\right)^4}=\frac{2^4-1^4}{2^4+1^4}=\frac{15}{17}.$$

f). $$\lim\frac{\left(\sqrt{n^2+1}+n\right)^2}{\sqrt[3]{n^6+1}}=\lim\frac{\left(n\sqrt{1+\frac{1}{n^2}}+n\right)^2}{\sqrt[3]{n^6\left(1+\frac{1}{n^6}\right)}}=\lim\frac{n^2\left(\sqrt{1+\frac{1}{n^2}}+1\right)^2}{n^2\sqrt[3]{1+\frac{1}{n^6}}}$$

$$=\lim\frac{\left(\sqrt{1+\frac{1}{n^2}}+1\right)^2}{\sqrt[3]{1+\frac{1}{n^6}}}=\frac{(1+1)^2}{1}=4.$$

Câu 17: Tìm các giới hạn sau:

a). $\lim\left(-2n^3+3n+5\right)$ b). $\lim\sqrt{2n^4+5n^3-7n}$ c). $\lim\sqrt[3]{1+2n-n^3}$

d). $\lim\left(3n+\cos n\right)$ e). $\lim\left(\frac{2}{3}n^2-3\sin n^3+5\right)$ f). $\lim\left(2n^2\cos n^2-4n^3\right)$

LỜI GIẢI

a). Ta có $L=\lim\left(-2n^3+3n+5\right)=\lim n^3\left(\frac{-2n^3+3n+5}{n^3}\right)=\lim n^3\left(-2+\frac{3}{n^2}+\frac{5}{n^3}\right)$.

Do $\lim \frac{3}{n^2}=0$ và $\lim \frac{5}{n^3}=0$ nên $\lim\left(-2+\frac{3}{n^2}+\frac{5}{n^3}\right)=-2$ (1), ngoài ra $\lim n^3=+\infty$ (2). Từ (1) và (2) có $L=-\infty$.

b). Ta có $L=\lim\sqrt{2n^4+5n^3-7n}=\lim\sqrt{n^4\left(\frac{2n^4+5n^3-7n}{n^4}\right)}=\lim n^2\sqrt{2+\frac{5}{n}-\frac{7}{n^3}}$

Do $\lim\frac{5}{n}=0$, $\lim\frac{7}{n^3}=0$ nên $\lim\sqrt{2+\frac{5}{n}-\frac{7}{n^3}}=2$ (1) và $\lim n^2=+\infty$ (2). Từ (1) và (2) suy ra $L=+\infty$.

c). Ta có $L=\lim\sqrt[3]{1+2n-n^3}=\lim\sqrt[3]{n^3\left(\frac{1+2n-n^3}{n^3}\right)}=\lim n.\sqrt[3]{\frac{1}{n^3}+\frac{2}{n^2}-1}$. Ta có $\lim\frac{1}{n^3}=0$, $\lim\frac{2}{n^2}=0$ nên $\lim\left(\sqrt[3]{\frac{1}{n^3}+\frac{2}{n^2}-1}\right)=-1$ (1) và $\lim n=+\infty$ (2). Từ (1) và (2) suy ra $L=-\infty$.

d). $L=\lim(3n+\cos n)=\lim\left[n\left(\frac{3n+\cos n}{n}\right)\right]=\lim\left[n\left(3+\frac{\cos n}{n}\right)\right]$.

Có $|\cos n|\le 1$ nên $\left|\frac{\cos n}{n}\right|\le\left|\frac{1}{n}\right|=\frac{1}{n}$ mà $\lim\frac{1}{n}=0$ nên $\lim\frac{\cos n}{n}=0$ (1) và $\lim n=+\infty$ (2). Từ (1) va (2) suy ra $L=+\infty$.

e). $L=\lim n^2\left(\frac{\frac{2}{3}n^2-3\sin n^3+5}{n^2}\right)=\lim n^2\left(\frac{2}{3}-3.\frac{\sin n^3}{n^2}+\frac{5}{n^2}\right)$. Có $\lim\frac{5}{n^2}=0$, có $\left|\frac{\sin n^3}{n^2}\right|\le\frac{1}{n^2}$ mà $\lim\frac{1}{n^2}=0$ nên $\lim\frac{\sin n^3}{n^2}=0$, do đó $\lim\left(\frac{2}{3}-3.\frac{\sin n^3}{n^2}+\frac{5}{n^2}\right)=\frac{2}{3}$ (1) ngoài ra $\lim n^2=+\infty$ (2). Từ (1) và (2) có $L=+\infty$.

f). $L=\lim\left(2n^2\cos n^2-4n^3\right)=\lim n^3\left(\frac{2n^2\cos n^2-4n^3}{n^3}\right)=\lim n^3\left(2.\frac{\cos n^2}{n}-4\right)$. Ta có $\left|\frac{\cos n^2}{n}\right|\le\left|\frac{1}{n}\right|=\frac{1}{n}$ mà $\lim\frac{1}{n}=0\Rightarrow\lim\frac{\cos n^2}{n}=0$ do đó $\lim\left(2.\frac{\cos n^2}{n}-4\right)=-4$ (1), ngoài ra $\lim n^3=+\infty$ (2). Từ (1) và (2) có $L=-\infty$.

Câu 18: Tìm các giới hạn sau:

a). $\lim\frac{n^5+n^4-3n-2}{4n^3+6n^2+9}$ b). $\lim\frac{-2n^3+3n-2}{4n+5}$ c). $\lim\frac{\sqrt[3]{n^6-7n^3-5n+8}}{n+2}$

d). $\lim\frac{n\sqrt{2n^2-1}}{\sqrt[3]{n^2+2n}}$ e). $\lim\left(\sqrt{n^2+n+1}-\sqrt{n+3}\right)$ f). $\lim\left(2^{n+3}-3^{n-2}\right)$.

LỜI GIẢI.

a). $L=\lim\frac{n^5+n^4-3n-2}{4n^3+6n^2+9}=\lim\frac{\frac{n^5+n^4-3n-2}{n^5}}{\frac{4n^3+6n^2+9}{n^5}}=\lim\left(n.\frac{1+\frac{1}{n}-\frac{3}{n^2}-\frac{2}{n^5}}{4+\frac{6}{n^3}+\frac{9}{n^5}}\right)$

Ta có $\lim\left(\frac{1+\frac{1}{n}-\frac{3}{n^2}-\frac{2}{n^5}}{4+\frac{6}{n^3}+\frac{9}{n^5}}\right)=\frac{1}{4}$ và $\lim n=+\infty$. Do đó $L=+\infty$.

b). Tương tự câu a).

c). $L=\lim\frac{\sqrt[3]{n^6-7n^3-5n+8}}{n+2}=\lim\frac{\sqrt[3]{n^6.\frac{n^6-7n^3-5n+8}{n^6}}}{n+2}=\lim\frac{n^3.\sqrt[3]{1-\frac{7}{n^3}-\frac{5}{n^5}+\frac{8}{n^6}}}{n+2}$

$=\lim\left(n^2.\frac{\sqrt[3]{1-\frac{7}{n^3}-\frac{5}{n^5}+\frac{8}{n^6}}}{1+\frac{2}{n}}\right).$

Ta có $\lim\left(\frac{\sqrt[3]{1-\frac{7}{n^3}-\frac{5}{n^5}+\frac{8}{n^6}}}{1+\frac{2}{n}}\right)=1$ và $\lim n^2=+\infty$, từ đó suy ra $L=+\infty$.

d). $L=\lim\frac{n\sqrt{2n^2-1}}{\sqrt[3]{n^2+2n}}$

$=\lim\frac{n^2.\sqrt{2-\frac{1}{n^2}}}{\sqrt[3]{n^2\left(\frac{n^2+2n}{n^2}\right)}}=\lim\frac{\left(\sqrt[3]{n^2}\right)^3.\sqrt{2-\frac{1}{n^2}}}{\sqrt[3]{n^2}.\sqrt[3]{1+\frac{2}{n^2}}}=\lim\left[\left(\sqrt[3]{n^2}\right)^2.\frac{\sqrt{2-\frac{1}{n^2}}}{\sqrt[3]{1+\frac{2}{n^2}}}\right].$

Do $\lim\left[\frac{\sqrt{2-\frac{1}{n^2}}}{\sqrt[3]{1+\frac{2}{n^2}}}\right]=\sqrt{2}$ và $\lim\left(\sqrt[3]{n^2}\right)^2=+\infty$ nên $L=+\infty$.

e). $L=\lim\left(\sqrt{n^2+n+1}-\sqrt{n+3}\right)=\lim\frac{n^2-2}{\sqrt{n^2+n+1}+\sqrt{n+3}}=\lim\left(n.\frac{1-\frac{2}{n^2}}{\sqrt{1+\frac{1}{n}+\frac{1}{n^2}}+\sqrt{\frac{1}{n}+\frac{3}{n^2}}}\right)$

Do $\lim\frac{1-\frac{2}{n^2}}{\sqrt{1+\frac{1}{n}+\frac{1}{n^2}}+\sqrt{\frac{1}{n}+\frac{3}{n^2}}}=1$ và $\lim n=+\infty$ nên $L=+\infty$.

f). $L = \lim\left(2^{n+3} - 3^{n-2}\right) = \lim\left(8.2^n - \frac{3^n}{9}\right) = \lim 3^n\left(8.\left(\frac{2}{3}\right)^n - \frac{1}{9}\right)$. Do $\lim\left(\frac{2}{3}\right)^n = 0$ nên $\lim\left(8.\left(\frac{2}{3}\right)^n - \frac{1}{9}\right) = -\frac{1}{9}$ ngoài ra $\lim 3^n = +\infty$. Vậy $L = -\infty$.