DẠNG 3: Chứng minh đẳng thức:

Bài 11:

a). Cho hàm số $y = x\sin x$. Chứng minh $x.y'' - 2\left(y' - \sin x\right) + xy = 0 \quad (*)$

b). Cho hàm số : $y = \sqrt{2x - x^2}$ chứng minh: $y^3.y'' + 1 = 0 \quad (*)$

c). Cho hàm số: $y = x\tan x$ chứng minh: $x^2.y'' - 2\left(x^2 + y^2\right)\left(1 + y\right) = 0 \quad (*)$

d). Cho hàm số: $y = \dfrac{x-3}{x+4}$ chứng minh: $2\left(y'\right)^2 = \left(y-1\right).y'' \quad (*)$

LỜI GIẢI

a). Cho hàm số $y = x\sin x$. Chứng minh $x.y'' - 2\left(y' - \sin x\right) + xy = 0 \quad (*)$

Ta có $y' = \left(x\sin x\right)' \Leftrightarrow y' = x'.\sin x + x.\left(\sin x\right)' \Leftrightarrow y' = \sin x + x\cos x$

$y'' = \left(\sin x + x\cos x\right)' = \left(\sin x\right)' + \left(x\cos x\right)' = \cos x + x'.\cos x + x.\left(\cos x\right)' = 2\cos x - x\sin x$

$(1) \Leftrightarrow x\left(2\cos x - x\sin x\right) - 2\left(\sin x + x\cos x - \sin x\right) + x^2\sin x = 0$

$\Leftrightarrow 2x\cos x - x^2\sin x - 2x\cos x + x^2\sin x = 0 \Leftrightarrow 0 = 0$ (đpcm).

b). Cho hàm số : $y = \sqrt{2x - x^2}$ chứng minh: $y^3.y'' + 1 = 0 \quad (*)$

Ta có: $y' = \left(\sqrt{2x - x^2}\right)' \Leftrightarrow y' = \dfrac{1}{2\sqrt{2x - x^2}}.\left(2x - x^2\right)' = \dfrac{1-x}{\sqrt{2x - x^2}}.$

$$y'' = \frac{\left(1-x\right)'.\sqrt{2x - x^2} - \left(\sqrt{2x - x^2}\right)'.\left(1-x\right)}{\left(\sqrt{2x - x^2}\right)^2} = \frac{-\sqrt{2x - x^2} - \dfrac{1-x}{\sqrt{2x - x^2}}.\left(1-x\right)}{\left(\sqrt{2x - x^2}\right)^2}$$

$$= \frac{-\left(2x - x^2\right) - \left(1-x\right)^2}{\sqrt{2x - x^2}.\left(\sqrt{2x - x^2}\right)^2} = \frac{-1}{\left(\sqrt{2x - x^2}\right)^3}.$$

$$(*) \Leftrightarrow \left(\sqrt{2x - x^2}\right)^3.\frac{-1}{\left(\sqrt{2x - x^2}\right)^3} + 1 = 0 \Leftrightarrow -1 + 1 = 0 \text{ (đpcm).}$$

c). Cho hàm số: $y = x\tan x$ chứng minh: $x^2.y'' - 2\left(x^2 + y^2\right)\left(1 + y\right) = 0 \quad (*)$

Ta có: $y' = \left(x\tan x\right)' = x'.\tan x + x.\left(\tan x\right)' = \tan x + x\left(1 + \tan^2 x\right)$

$y'' = \left(\tan x\right)' + x'.\left(1 + \tan x\right) + x.\left(1 + \tan x\right)' = 2\left(1 + \tan^2 x\right) + x.\left(2\tan x\right).\left(\tan^2 + 1\right)$

$= 2\left(1 + \tan^2 x\right)\left(1 + x\tan x\right)$

$(*) \Leftrightarrow 2x^2\left(1 + \tan^2 x\right).\left(1 + x\tan x\right) - 2\left(x^2 + x^2\tan^2 x\right)\left(1 + x\tan x\right) = 0$

$\Leftrightarrow 2x^2\left(1 + \tan^2 x\right)\left(1 + x\tan x\right) - 2x^2\left(1 + \tan^2 x\right)\left(1 + x\tan x\right) = 0$

$\Leftrightarrow 0 = 0$ (đpcm).

d). Cho hàm số: $y = \dfrac{x-3}{x+4}$ chứng minh: $2\left(y'\right)^2 = \left(y-1\right).y'' \quad (*)$

Ta có: $y' = \left(\dfrac{x-3}{x+4}\right)' = \dfrac{7}{\left(x+4\right)^2}$

$y'' = \dfrac{-7\left((x+4)^2\right)'}{(x+4)^4} = \dfrac{-14}{(x+4)^3}$

$(*) \Leftrightarrow 2\left(\dfrac{7}{(x+4)^2}\right)^2 = \left(\dfrac{x-3}{x+4}-1\right).\left(\dfrac{-14}{(x+4)^3}\right) \Leftrightarrow \dfrac{98}{(x+4)^4} = \dfrac{98}{(x+4)^4}$ (đpcm).

e) Cho hàm số $y = \cos^2 3x$ chứng minh: $18(2y-1)+y''=0$ $(*)$

Ta có: $y = \cos^2 3x$

$y' = 2.\cos 3x(\cos 3x)' = 2\cos 3x.(-\sin 3x)(3x)' = -3\sin 6x$

$y'' = -18\cos 6x$

$(*) \Leftrightarrow 18(2\cos^2 3x-1)-18\cos 6x = 0 \Leftrightarrow 18.\cos 6x - 18\cos 6x = 0$ (đpcm).

Bài 12:

a).Cho hàm số $y = \dfrac{\sin^3 x+\cos^3 x}{1-\sin x.\cos x}$. Chứng minh $y''+y=0$ $(*)$

b). Cho hàm số $y = (x^2-1)^2$. Chứng minh: $y^4 + 2xy'''-4y''=40$ $(*)$

c). Cho hàm số $y = \sqrt{x+\sqrt{1+x^2}}$. Chứng minh: $4(x^2+1).y''+4x.y'-y=0$ $(*)$

d). Chứng minh $(1+x^2).y''+x.y'-k^2.y=0$ nếu $y = \left(x+\sqrt{x^2+1}\right)^k$

LỜI GIẢI

a).Cho hàm số $y = \dfrac{\sin^3 x+\cos^3 x}{1-\sin x.\cos x}$ chứng minh $y''+y=0$ $(*)$

Ta có: $y = \dfrac{(\sin x+\cos x)(\sin^2 x+\cos^2 x-\sin x\cos x)}{1-\sin x\cos x}$

$= \dfrac{(\sin x+\cos x)(1-\sin x\cos x)}{1-\sin x\cos x} = \sin x+\cos x$

$y' = \cos x-\sin x$

$y'' = -\sin x-\cos x$

$(*) \Leftrightarrow -\sin x-\cos x+\sin x+\cos x = 0 \Leftrightarrow 0=0$ (đpcm).

b). Cho hàm số $y = (x^2-1)^2$. Chứng minh: $y^4 + 2xy'''-4y''=40$ $(*)$

Ta có: $y = x^4-2x^2+1$

$y' = 4x^3-4x$

$y'' = 12x^2-4$

$y''' = 24x$

$y'''' = 24.$

$(*) \Leftrightarrow 24+2x(24x)-4(12x^2-4)=40.$

$\Leftrightarrow 24+48x^2-48x^2+16=40 \Leftrightarrow 40=40$ (đpcm).

c). Cho hàm số $y = \sqrt{x+\sqrt{1+x^2}}$. Chứng minh: $4(x^2+1).y''+4x.y'-y=0$ $(*)$

Ta có: $y' = \frac{1}{2\sqrt{x+\sqrt{1+x^2}}}.\left(1+\frac{x}{\sqrt{1+x^2}}\right) = \frac{\sqrt{x+\sqrt{1+x^2}}}{2\sqrt{1+x^2}}$

$$y'' = \frac{\left(\sqrt{x+\sqrt{1+x^2}}\right)'.2\sqrt{1+x^2} - \left(2\sqrt{1+x^2}\right)'.\sqrt{x+\sqrt{1+x^2}}}{\left(2\sqrt{1+x^2}\right)^2}$$

$$= \frac{\sqrt{x+\sqrt{1+x^2}}.\left(2\sqrt{1+x^2}-4x\right)}{8\left(1+x^2\right).\sqrt{1+x^2}}$$

$$(*) \Leftrightarrow 4\left(x^2+1\right)\frac{\sqrt{x+\sqrt{1+x^2}}\left(2\sqrt{1+x^2}-4x\right)}{8\left(1+x^2\right)\sqrt{1+x^2}} + 4x.\frac{\sqrt{x+\sqrt{1+x^2}}}{2\sqrt{1+x^2}} - \sqrt{x+\sqrt{1+x^2}} = 0$$

$$\Leftrightarrow \frac{\sqrt{x+\sqrt{1+x^2}}.\left(2\sqrt{1+x^2}-4x\right)}{2\sqrt{1+x^2}} + 2x\frac{\sqrt{x+\sqrt{1+x^2}}}{\sqrt{1+x^2}} - \sqrt{x+\sqrt{1+x^2}} = 0$$

$$\Leftrightarrow \sqrt{x+\sqrt{1+x^2}} - \frac{2x\sqrt{x+\sqrt{1+x^2}}}{\sqrt{1+x^2}} + \frac{2x\sqrt{x+\sqrt{1+x^2}}}{\sqrt{1+x^2}} - \sqrt{x+\sqrt{1+x^2}} = 0$$

$\Leftrightarrow 0 = 0$ (đpcm).

d). Chứng minh $\left(1+x^2\right).y''+x.y'-k^2.y=0$ nếu $y = \left(x+\sqrt{x^2+1}\right)^k$

Ta có: $y = \left(x+\sqrt{x^2+1}\right)^k \Rightarrow y' = k\left(x+\sqrt{x^2+1}\right)^{k-1}.\left(1+\frac{x}{\sqrt{x^2+1}}\right)$

$$= k\left(x+\sqrt{x^2+1}\right)^{k-1}.\left(\frac{x+\sqrt{x^2+1}}{\sqrt{x^2+1}}\right) = k.\frac{\left(x+\sqrt{x^2+1}\right)^2}{\sqrt{x^2+1}}$$

$$y'' = k.\frac{\left[\left(x+\sqrt{x^2+1}\right)^k\right]'.\sqrt{x^2+1} - \left(\sqrt{x^2+1}\right)'.\left(x+\sqrt{x^2+1}\right)^k}{x^2+1}$$

$$= k.\frac{\frac{k\left(x+\sqrt{x^2+1}\right)^k}{\sqrt{x^2+1}}.\sqrt{x^2+1} - \frac{x}{\sqrt{x^2+1}}.\left(x+\sqrt{x^2+1}\right)^k}{\left(x^2+1\right)}$$

$$= \frac{k\left(x+\sqrt{x^2+1}\right)^k\left(k\sqrt{x^2+1}-x\right)}{\left(x^2+1\right)\sqrt{x^2+1}}$$

$$(*) \Leftrightarrow \left(1+x^2\right)\frac{k\left(x+\sqrt{x^2+1}\right)^k.\left(k\sqrt{x^2+1}-x\right)}{\left(x^2+1\right)\sqrt{x^2+1}} + \frac{x.k\left(x+\sqrt{1+x^2}\right)^k}{\sqrt{1+x^2}}$$

$$\Leftrightarrow \frac{k\left(x+\sqrt{x^2+1}\right)^k.\left(k\sqrt{x^2+1}-x\right)}{\sqrt{x^2+1}} + \frac{x.k\left(x^2+\sqrt{1+x^2}\right)^k}{\sqrt{1+x^2}} - k^2\left(x+\sqrt{x^2+1}\right)^k = 0$$

Quy đồng đặt thừa số chung được:

$$\frac{\left(x+\sqrt{x^2+1}\right)^k}{\sqrt{x^2+1}}\left(k^2\sqrt{x^2+1}-kx+kx-k^2\sqrt{x^2+1}\right)=0 \Leftrightarrow 0=0 \text{ (đpcm).}$$

Ví dụ 3: Chứng minh rằng với mọi số nguyên $n \geq 1$ ta có:

a)Nếu $y=\frac{1}{x}$ thì $y^n=(-1)^n.\frac{n!}{x^{n+1}}$.

b) Nếu $y=\cos x$ thì $y^{4n}=\cos x$.

Ví dụ 4: Chứng minh rằng:

a)Nếu $y=\sin ax$ thì $y^{4n}=a^{4n}.\sin ax$ (a là hằng số).

b) Nếu $y=\sin^2 x$ thì $y^{4n}=-2^{4n-1}\cos 2x$.