Câu 4: Giải các phương trình lượng giác sau:

1). $\sin^4 + \cos^4 x - 2\sin 2x + \frac{3}{4}\sin^2 2x = 0$ 2). $\cos^6 2x + \sin^6 2x = \frac{15}{8}\cos 4x - \frac{1}{2}$

3). $\sin^4 x + \frac{5}{3}\cos^4 x = 1$ 4). $\sin^4 x + \cos^4 x + \cos\left(x - \frac{\pi}{4}\right)\sin\left(x - \frac{3\pi}{4}\right) = \frac{1}{2}$ 5). $\sin^4 x + \cos 2x + 4\sin^6 x = 0$ 6). $\cos 8x + \sin^3 x\cos x - \cos^3 x\sin x - 1 = 0$

LỜI GIẢI

1). $\sin^4 + \cos^4 x - 2\sin 2x + \frac{3}{4}\sin^2 2x = 0\ (1)$

$\Leftrightarrow \left(1 - \frac{1}{2}\sin^2 2x\right) - 2\sin 2x + \frac{3}{4}\sin^2 2x = 0 \Leftrightarrow \frac{1}{4}\sin^2 2x - 2\sin 2x + 1 = 0\ (1')$

Đặt $\sin 2x = t, t \in [-1;1]$. Phương trình (1') trở thành: $\frac{1}{4}t^2 - 2t + 1 = 0$

$\Leftrightarrow t = 4 + 2\sqrt{3} \vee t = 4 - 2\sqrt{3}$. So với điều kiện nhận $t = 4 - 2\sqrt{3} \Rightarrow \sin 2x = 4 - 2\sqrt{3}$

$$\Leftrightarrow \left[\begin{array}{l} 2x = \arcsin\left(4 - 2\sqrt{3}\right) + k2\pi \\ 2x = \pi - \arcsin\left(4 - 2\sqrt{3}\right) + k2\pi \end{array}\right. \Leftrightarrow \left[\begin{array}{l} x = \dfrac{\arcsin\left(4 - 2\sqrt{3}\right)}{2} + k\pi, (k \in \mathbb{Z}) \\ x = \dfrac{\pi - \arcsin\left(4 - 2\sqrt{3}\right)}{2} + k\pi, (k \in \mathbb{Z}) \end{array}\right.$$

Vậy nghiệm của phương trình: $x = \dfrac{\arcsin\left(4 - 2\sqrt{3}\right)}{2} + k\pi, x = \dfrac{\pi - \arcsin\left(4 - 2\sqrt{3}\right)}{2} + k\pi, (k \in \mathbb{Z})$.

2). $\cos^6 2x + \sin^6 2x = \frac{15}{8}\cos 4x - \frac{1}{2}\ (1)$

$(1) \Leftrightarrow 1 - \frac{3}{4}\sin^2 2x = \frac{15}{8}\cos 4x - \frac{1}{2} \Leftrightarrow 1 - \frac{3}{4}\cdot\frac{1 - \cos 4x}{2} = \frac{15}{8}\cos 4x - \frac{1}{2}$

$\Leftrightarrow 8 - 3(1 - \cos 4x) = 15\cos 4x - 4 \Leftrightarrow \cos 4x = \frac{3}{4}$

$\Leftrightarrow 4x = \pm\arccos\frac{3}{4} + k2\pi \Leftrightarrow x = \pm\frac{1}{4}\arccos\frac{3}{4} + \frac{k\pi}{2} (k \in \mathbb{Z})$

Vậy nghiệm của phương trình: $x = \pm\frac{1}{4}\arccos\frac{3}{4} + \frac{k\pi}{2} (k \in \mathbb{Z})$.

3). $\sin^4 x + \frac{5}{3}\cos^4 x = 1 \Leftrightarrow \left(\sin^2 x\right)^2 + \frac{5}{3}\left(\cos^2 x\right)^2 = 1$

$\Leftrightarrow \left(\frac{1 - \cos 2x}{2}\right)^2 + \frac{5}{3}\left(\frac{1 + \cos 2x}{2}\right)^2 = 1\ (1)$. Đặt $\cos 2x = t, t \in [-1;1]$

$(1) \Leftrightarrow \frac{1 - 2t + t^2}{4} + \frac{5}{3}\cdot\frac{1 + 2t + t^2}{4} = 1 \Leftrightarrow 8t^2 + 4t - 4 = 0 \Leftrightarrow t = -1 \vee t = \frac{1}{2}$.

Với $t = -1 \Leftrightarrow \cos 2x = -1 \Leftrightarrow 2x = \pi + k2\pi \Leftrightarrow x = \frac{\pi}{2} + k\pi, (k \in \mathbb{Z})$

Với $t = \frac{1}{2} \Leftrightarrow \cos 2x = \frac{1}{2} \Leftrightarrow \left[\begin{array}{l} 2x = \frac{\pi}{3} + k2\pi \\ 2x = -\frac{\pi}{3} + k2\pi \end{array}\right. \Leftrightarrow \left[\begin{array}{l} x = \frac{\pi}{6} + k\pi \\ x = -\frac{\pi}{6} + k\pi \end{array}\right., (k \in \mathbb{Z})$

Vậy nghiệm của phương trình: $x = \frac{\pi}{2} + k\pi,\ x = \frac{\pi}{6} + k\pi, x = -\frac{\pi}{6} + k\pi, (k \in \mathbb{Z})$

4). $\sin^4 x+\cos^4 x+\cos\left(x-\frac{\pi}{4}\right)\sin\left(x-\frac{3\pi}{4}\right)=\frac{1}{2}$

$\Leftrightarrow\left(1-\frac{1}{2}\sin^2 2x\right)+\frac{1}{2}\left[\sin\left(-\frac{\pi}{2}\right)+\sin(2x-\pi)\right]=\frac{1}{2}\Leftrightarrow 1-\frac{1}{2}\sin^2 2x+\frac{1}{2}(-1-\sin 2x)=\frac{1}{2}$

$\Leftrightarrow \sin^2 2x+\sin 2x=0\Leftrightarrow \sin 2x=0\vee \sin 2x=-1$

Với $\sin 2x=0\Leftrightarrow 2x=k\pi\Leftrightarrow x=\frac{k\pi}{2},(k\in\mathbb{Z})$

Với $\sin 2x=-1\Leftrightarrow 2x=-\frac{\pi}{2}+k2\pi\Leftrightarrow x=-\frac{\pi}{4}+k\pi,(k\in\mathbb{Z})$.

Vậy nghiệm của phương trình: $x=\frac{k\pi}{2}$, $x=-\frac{\pi}{4}+k\pi,(k\in\mathbb{Z})$

5). $\sin^4 x+\cos 2x+4\sin^6 x=0\Leftrightarrow\left(\sin^2 x\right)^2+\cos 2x+4\left(\sin^2 x\right)^3=0$

$\Leftrightarrow\left(\frac{1-\cos 2x}{2}\right)^2+\cos 2x+4\left(\frac{1-\cos 2x}{2}\right)^3=0$ (1). Đặt $\cos 2x=t,t\in[-1;1]$. Phương trình (1) trở thành:

$\left(\frac{1-t}{2}\right)^2+t+4\left(\frac{1-t}{2}\right)^3=0\Leftrightarrow 2t^3-7t^2+4t-3=0\Leftrightarrow t=3$ (loại).

Kết luận phương trình vô nghiệm.

6). $\cos 8x+\sin^3 x\cos x-\cos^3 x\sin x-1=0$ (1)

$(1)\Leftrightarrow \cos 8x+\sin x\cos x\left(\sin^2 x-\cos^2 x\right)-1=0\Leftrightarrow \cos 8x-\frac{1}{2}\sin 2x\cos 2x-1=0$

$\Leftrightarrow 1-2\sin^2 4x-\frac{1}{4}\sin 4x-1=0\Leftrightarrow 2\sin^2 4x+\frac{1}{4}\sin 4x=0\Leftrightarrow \sin 4x=0\vee \sin 4x=-\frac{1}{8}$

Với $\sin 4x=0\Leftrightarrow 4x=k\pi\Leftrightarrow x=\frac{k\pi}{4},(k\in\mathbb{Z})$.

Với $\sin 4x=-\frac{1}{8}\Leftrightarrow\left[\begin{array}{l}4x=\arcsin\left(-\frac{1}{8}\right)+k2\pi\\4x=\pi-\arcsin\left(-\frac{1}{8}\right)+k2\pi\end{array}\right.\Leftrightarrow\left[\begin{array}{l}x=\frac{1}{4}\arcsin\left(-\frac{1}{8}\right)+\frac{k\pi}{2}\\x=\frac{\pi}{4}-\frac{1}{4}\arcsin\left(-\frac{1}{8}\right)+\frac{k\pi}{2}\end{array}\right.,(k\in\mathbb{Z})$

Vậy nghiệm của phương trình: $x=\frac{k\pi}{4}$, $x=\frac{1}{4}\arcsin\left(-\frac{1}{8}\right)+\frac{k\pi}{2}$, $x=\frac{\pi}{4}-\frac{1}{4}\arcsin\left(-\frac{1}{8}\right)+\frac{k\pi}{2},(k\in\mathbb{Z})$

Câu 5: Giải các phương trình lượng giác sau:

1). $\left(\sin 2x+\sqrt{3}\cos 2x\right)^2=2\cos\left(2x-\frac{\pi}{6}\right)$

2). $\cot x+\left(1+\tan x\tan\frac{x}{2}\right)\sin x=4$

3). $\cot x-\tan x=\frac{2}{\sin 2x}-4\sin 2x$

4). $\sqrt{2}(2\sin x-1)=4(\sin x-1)-\cos\left(2x+\frac{\pi}{4}\right)-\sin\left(2x+\frac{\pi}{4}\right)$

5). $\cot x-1=\frac{\cos 2x}{1+\tan x}+\sin^2 x-\frac{1}{2}\sin 2x$ (ĐH khối A 2003)

6). $5\sin x-2=3(1-\sin x)\tan^2 x$ (ĐH khối B 2004)

7). $\cos^2 3x\cos 2x-\cos^2 x=0$ (ĐH khối A 2005).

8). $\sin^4 x+\cos^4 x+\cos\left(x-\frac{\pi}{4}\right)\sin\left(3x-\frac{\pi}{4}\right)-\frac{3}{2}=0$ (ĐH khối D 2005).

9). $\frac{(1+\sin x+\cos 2x)\sin\left(x+\frac{\pi}{4}\right)}{1+\tan x}=\frac{1}{\sqrt{2}}\cos x$ (ĐH khối A 2010).

10). $\frac{\sin^4 x+\cos^4 x}{5\sin 2x}=\frac{1}{2}\cot 2x-\frac{1}{8\sin 2x}$ (1) [Dự bị 2 ĐH02]

11). $\cot x-\tan x+4\sin 2x=\frac{2}{\sin 2x}$.[ĐH B03]

12). $3\cos 4x-8\cos^6 x+2\cos^2 x+3=0$ (1) [Dự bị 1 ĐH B03]

13). $\cot x=\tan x+\frac{2\cos 4x}{\sin 2x}$ [Dự bị 2 ĐH D03]

LỜI GIẢI

1). $\left(\sin 2x+\sqrt{3}\cos 2x\right)^2=2\cos\left(2x-\frac{\pi}{6}\right)$ (1)

$(1)\Leftrightarrow 4\left(\frac{1}{2}\sin 2x+\frac{\sqrt{3}}{2}\cos 2x\right)^2=2\cos\left(2x-\frac{\pi}{6}\right)$

$\Leftrightarrow 4\left(\cos 2x\cos\frac{\pi}{6}+\sin 2x\sin\frac{\pi}{6}\right)^2=2\cos\left(2x-\frac{\pi}{6}\right)$

$\Leftrightarrow 2\cos^2\left(2x-\frac{\pi}{6}\right)-\cos\left(2x-\frac{\pi}{6}\right)=0\Leftrightarrow\cos\left(2x-\frac{\pi}{6}\right)=0\vee\cos\left(2x-\frac{\pi}{6}\right)=\frac{1}{2}$

Với $\cos\left(2x-\frac{\pi}{6}\right)=0\Leftrightarrow 2x-\frac{\pi}{6}=\frac{\pi}{2}+k\pi\Leftrightarrow x=\frac{\pi}{3}+\frac{k\pi}{2},(k\in\mathbb{Z})$

Với $\cos\left(2x-\frac{\pi}{6}\right)=\frac{1}{2}\Leftrightarrow\left[\begin{array}{l}2x-\frac{\pi}{6}=\frac{\pi}{3}+k2\pi\\2x-\frac{\pi}{6}=-\frac{\pi}{3}+k2\pi\end{array}\right.\Leftrightarrow\left[\begin{array}{l}x=\frac{\pi}{4}+k\pi\\x=-\frac{\pi}{12}+k\pi\end{array}\right.,(k\in\mathbb{Z})$

Vậy nghiệm của phương trình: $x=\frac{\pi}{3}+\frac{k\pi}{2}$, $x=\frac{\pi}{4}+k\pi, x=-\frac{\pi}{12}+k\pi,(k\in\mathbb{Z})$

2). $\cot x+\left(1+\tan x\tan\frac{x}{2}\right)\sin x=4$ (1). Điều kiện $\begin{cases}\sin x\neq 0\\\cos x\neq 0\\\cos\frac{x}{2}\neq 0\end{cases}$

Ta có: $1+\tan x\tan\frac{x}{2}=1+\frac{\sin x\sin\frac{x}{2}}{\cos x\cos\frac{x}{2}}=\frac{\cos x\cos\frac{x}{2}+\sin x\sin\frac{x}{2}}{\cos x\cos\frac{x}{2}}=\frac{\cos\left(x-\frac{x}{2}\right)}{\cos x\cos\frac{x}{2}}=\frac{1}{\cos x}$

$(1)\Leftrightarrow\frac{\cos x}{\sin x}+\frac{\sin x}{\cos x}=4\Leftrightarrow\cos^2 x+\sin^2 x=4\sin x\cos x\Leftrightarrow 2\sin 2x=1$

$$\Leftrightarrow \sin 2x = \frac{1}{2} \Leftrightarrow \left[\begin{array}{l} 2x = \frac{\pi}{6} + k2\pi \\ 2x = \frac{5\pi}{6} + k2\pi \end{array}\right. \Leftrightarrow \left[\begin{array}{l} x = \frac{\pi}{12} + k\pi \\ x = \frac{5\pi}{12} + k\pi \end{array}\right., (k \in \mathbb{Z})$$

Vậy nghiệm của phương trình: $x = \frac{\pi}{12} + k\pi, x = \frac{5\pi}{12} + k\pi, (k \in \mathbb{Z})$

3). $\cot x - \tan x = \frac{2}{\sin 2x} - 4\sin 2x \quad (1)$. Điều kiện $\sin 2x \neq 0 \Leftrightarrow x \neq \frac{k\pi}{2}$

$(1) \Leftrightarrow \frac{\cos x}{\sin x} - \frac{\sin x}{\cos x} = \frac{2}{\sin 2x} - 4\sin 2x \quad \Leftrightarrow \frac{\cos^2 x - \sin^2 x}{\sin x \cos x} = \frac{2}{\sin 2x} - 4\sin 2x$

$\Leftrightarrow \frac{2\cos 2x}{\sin 2x} = \frac{2}{\sin 2x} - 4\sin 2x \quad \Leftrightarrow 2\cos 2x = 2 - 4\sin^2 2x$

$\Leftrightarrow \cos 2x = 1 - 2\left(1 - \cos^2 2x\right) \quad \Leftrightarrow 2\cos^2 2x - \cos 2x - 1 = 0$

$\Leftrightarrow \cos 2x = 1 \vee \cos 2x = -\frac{1}{2}$.

Với $\cos 2x = 1 \Leftrightarrow 2x = k2\pi \Leftrightarrow x = k\pi, (k \in \mathbb{Z})$ (loại)

Với $\cos 2x = -\frac{1}{2} \Leftrightarrow \cos 2x = \cos\frac{2\pi}{3} \Leftrightarrow \left[\begin{array}{l} 2x = \frac{2\pi}{3} + k2\pi \\ 2x = -\frac{2\pi}{3} + k2\pi \end{array}\right. \Leftrightarrow \left[\begin{array}{l} x = \frac{\pi}{3} + k\pi \\ x = -\frac{\pi}{3} + k\pi \end{array}\right., (k \in \mathbb{Z})$

Vậy nghiệm của phương trình: $x = \frac{\pi}{3} + k\pi, x = -\frac{\pi}{3} + k\pi, (k \in \mathbb{Z})$

4). $\sqrt{2}\left(2\sin x - 1\right) = 4\left(\sin x - 1\right) - \cos\left(2x + \frac{\pi}{4}\right) - \sin\left(2x + \frac{\pi}{4}\right) \quad (1)$

$\Leftrightarrow \sqrt{2}\left(2\sin x - 1\right) = 4\left(\sin x - 1\right) - \left[\cos\left(2x + \frac{\pi}{4}\right) + \sin\left(2x + \frac{\pi}{4}\right)\right]$

$\Leftrightarrow \sqrt{2}\left(2\sin x - 1\right) = 4\left(\sin x - 1\right) - \sqrt{2}\cos\left(2x + \frac{\pi}{4} - \frac{\pi}{4}\right)$

$\Leftrightarrow \sqrt{2}\left(2\sin x - 1\right) = 4\left(\sin x - 1\right) - \sqrt{2}\cos 2x$

$\Leftrightarrow \sqrt{2}\left(2\sin x - 1\right) = 4\left(\sin x - 1\right) - \sqrt{2}\left(1 - 2\sin^2 x\right)$

$\Leftrightarrow 2\sqrt{2}\sin^2 x + 2\left(2 - \sqrt{2}\right)\sin x - 4 = 0$

$\Leftrightarrow \sin x = 1 \vee \sin x = -\sqrt{2}$ (loại).

Với $\sin x = 1 \Leftrightarrow x = \frac{\pi}{2} + k2\pi, (k \in \mathbb{Z})$

Với Vậy nghiệm của phương trình: $x = \frac{\pi}{2} + k2\pi, (k \in \mathbb{Z})$