Ví dụ 2: Tính các giới hạn sau:

a). $\lim\limits_{x\to+\infty}(x+1)\sqrt{\dfrac{x}{2x^4+x^2+1}}$ b). $\lim\limits_{x\to-\infty}\dfrac{|x|+\sqrt{x^2+x}}{x+10}$ c). $\lim\limits_{x\to-\infty}\dfrac{\sqrt{x^2-3x}+2x}{3x-1}$

d). $\lim\limits_{x\to+\infty}\dfrac{\sqrt{x^2+x+2}+3x+1}{\sqrt{4x^2+1}+1-x}$ e). $\lim\limits_{x\to-\infty}\dfrac{\sqrt{x^2+x+2}+3x+1}{\sqrt{4x^2+1}+1-x}$

f). $\lim\limits_{x\to-\infty}\dfrac{\sqrt[3]{(x^3+2x^2)^2}+x\sqrt[3]{x^3+2x^2}+x^2}{3x^2-2x}$

LỜI GIẢI

a). $\lim\limits_{x\to+\infty}(x+1)\sqrt{\dfrac{x}{2x^4+x^2+1}}=\lim\limits_{x\to+\infty}\sqrt{\dfrac{x(x+1)^2}{2x^4+x^2+1}}=\lim\limits_{x\to+\infty}\sqrt{\dfrac{\frac{1}{x}+\frac{2}{x^2}+\frac{1}{x^3}}{2+\frac{1}{x^2}+\frac{1}{x^4}}}=0$. (Chú thích:

Vì $x\to+\infty$ nên $x>0\Rightarrow(x+1)>0$ do đó ta được được vào trong dấu căn.

b). $\lim\limits_{x\to-\infty}\dfrac{|x|+\sqrt{x^2+x}}{x+10}=\lim\limits_{x\to-\infty}\dfrac{|x|+\sqrt{x^2\left(1+\frac{1}{x}\right)}}{x+10}=\lim\limits_{x\to-\infty}\dfrac{|x|+|x|\sqrt{1+\frac{1}{x}}}{x+10}=\lim\limits_{x\to-\infty}\dfrac{-x-x\sqrt{1+\frac{1}{x}}}{x+10}=\lim\limits_{x\to-\infty}\dfrac{-1-\sqrt{1+\frac{1}{x}}}{1+\frac{10}{x}}=-2$.

(Chú giải: Vì $x\to-\infty$ nên $x<0$ do đó $|x|=-x$).

c). $\lim\limits_{x\to-\infty}\dfrac{\sqrt{x^2-3x}+2x}{3x-1}=\lim\limits_{x\to-\infty}\dfrac{\sqrt{x^2\left(1-\frac{3}{x}\right)}+2x}{3x-1}=\lim\limits_{x\to-\infty}\dfrac{|x|\sqrt{1-\frac{3}{x}}+2x}{3x-1}=\lim\limits_{x\to-\infty}\dfrac{-x\sqrt{1-\frac{3}{x}}+2x}{3x-1}=\lim\limits_{x\to-\infty}\dfrac{-\sqrt{1-\frac{3}{x}}+2}{3-\frac{1}{x}}$

$=\dfrac{-1+2}{3}=\dfrac{1}{3}$. (Chú giải: Vì $x\to-\infty$ nên $x<0$ do đó $|x|=-x$).

d). $\lim\limits_{x\to+\infty}\dfrac{\sqrt{x^2+x+2}+3x+1}{\sqrt{4x^2+1}+1-x}=\lim\limits_{x\to+\infty}\dfrac{\frac{\sqrt{x^2+x+2}+3x+1}{x}}{\frac{\sqrt{4x^2+1}+1-x}{x}}=\lim\limits_{x\to+\infty}\dfrac{\frac{\sqrt{x^2+x+2}}{x}+\frac{3x}{x}+\frac{1}{x}}{\frac{\sqrt{4x^2+1}}{x}+\frac{1}{x}-\frac{x}{x}}$

$=\lim\limits_{x\to+\infty}\dfrac{\sqrt{\frac{x^2+x+2}{x^2}}+3+\frac{1}{x}}{\sqrt{\frac{4x^2+1}{x^2}}+\frac{1}{x}-1}=\lim\limits_{x\to+\infty}\dfrac{\sqrt{1+\frac{1}{x}+\frac{2}{x^2}}+3+\frac{1}{x}}{\sqrt{4+\frac{1}{x^2}}+\frac{1}{x}-1}=\dfrac{1+3}{2-1}=4$.

e). $\lim\limits_{x\to-\infty}\dfrac{\sqrt{x^2+x+2}+3x+1}{\sqrt{4x^2+1}+1-x}=\lim\limits_{x\to-\infty}\dfrac{\frac{\sqrt{x^2+x+2}+3x+1}{x}}{\frac{\sqrt{4x^2+1}+1-x}{x}}=\lim\limits_{x\to-\infty}\dfrac{\frac{\sqrt{x^2+x+2}}{x}+\frac{3x}{x}+\frac{1}{x}}{\frac{\sqrt{4x^2+1}}{x}+\frac{1}{x}-\frac{x}{x}}$

$=\lim\limits_{x\to-\infty}\dfrac{-\sqrt{\frac{x^2+x+2}{x^2}}+3+\frac{1}{x}}{-\sqrt{\frac{4x^2+1}{x^2}}+\frac{1}{x}-1}=\lim\limits_{x\to-\infty}\dfrac{-\sqrt{1+\frac{1}{x}+\frac{2}{x^2}}+3+\frac{1}{x}}{-\sqrt{4+\frac{1}{x^2}}+\frac{1}{x}-1}=\dfrac{-1+3}{-2-1}=-\dfrac{2}{3}$

f). $\lim\limits_{x\to-\infty}\dfrac{\sqrt[3]{(x^3+2x^2)^2}+x\sqrt[3]{x^3+2x^2}+x^2}{3x^2-2x}=\lim\limits_{x\to-\infty}\dfrac{\sqrt[3]{\left[x^3\left(1+\frac{2}{x}\right)\right]^2}+\sqrt[3]{x^3\left(1+\frac{2}{x}\right)}+x^2}{3x^2-2x}$

$=\lim\limits_{x\to-\infty}\dfrac{\sqrt[3]{\left(x^2\right)^3\left(1+\frac{2}{x}\right)^2}+\sqrt[3]{x^3\left(1+\frac{2}{x}\right)}+x^2}{3x^2-2x}=\lim\limits_{x\to-\infty}\dfrac{x^2.\sqrt[3]{\left(1+\frac{2}{x}\right)^2}+x^2.\sqrt[3]{1+\frac{2}{x}}+x^2}{3x^2-2x}$

$=\lim\limits_{x\to-\infty}\dfrac{x^2\left[\sqrt[3]{\left(1+\frac{2}{x}\right)^2}+\sqrt[3]{1+\frac{2}{x}}+1\right]}{x^2\left(3-\frac{2}{x}\right)}=\lim\limits_{x\to-\infty}\dfrac{\sqrt[3]{\left(1+\frac{2}{x}\right)^2}+\sqrt[3]{1+\frac{2}{x}}+1}{3-\frac{2}{x}}=\dfrac{1+1+1}{3}=1.$

Ví dụ 3: Tìm các giới hạn sau:

a). $\lim\limits_{x\to\infty}x^2\left(\sqrt{\dfrac{x+2}{x}}-\sqrt[3]{\dfrac{x+3}{x}}\right)$ b). $\lim\limits_{x\to-\infty}\left[\sqrt{\dfrac{4x^4+1}{x+2x^4}}-\dfrac{\sqrt{2x^2-4}}{x}\right]$

LỜI GIẢI

a). Đặt $x=\dfrac{1}{y}$ khi $x\to\infty$ thì $y\to0$

$I=\lim\limits_{y\to0}\dfrac{\sqrt{1+2y}-\sqrt[3]{1+3y}}{y^2}=\lim\limits_{y\to0}\left[\dfrac{\sqrt{1+2y}-(1+y)}{y^2}-\dfrac{\sqrt[3]{1+3y}-(1+y)}{y^2}\right]$

$=\lim\limits_{y\to0}\left[\dfrac{-y^2}{y^2\sqrt{1+2y}+(1+y)}+\dfrac{y^2(y+3)}{y^2(\sqrt[3]{(1+3y)^2}+(1+y)\sqrt[3]{1+3y}+(1+y)^2)}\right]$

$=\lim\limits_{y\to0}\left[-\dfrac{1}{1+y+\sqrt{1+2y}}+\dfrac{y+3}{(1+y)^2+(1+y)\sqrt[3]{1+3y}+\sqrt[3]{(1+3y)^2}}\right]$

$=-\dfrac{1}{2}+1=\dfrac{1}{2}$. Vậy $I=\dfrac{1}{2}$

b). $\lim\limits_{x\to-\infty}\left[\sqrt{\dfrac{4x^4+1}{x+2x^4}}-\dfrac{\sqrt{2x^2-4}}{x}\right]=\lim\limits_{x\to-\infty}\left[\sqrt{\dfrac{\frac{4x^4+1}{x^4}}{\frac{x+2x^4}{x^4}}}-\dfrac{\sqrt{x^2\left(\frac{2x^2-4}{x^2}\right)}}{x}\right]$

$=\lim\limits_{x\to-\infty}\left(\sqrt{\dfrac{4+\frac{1}{x^4}}{\frac{1}{x^3}+2}}-\dfrac{|x|\sqrt{2-\frac{4}{x^2}}}{x}\right)=\lim\limits_{x\to-\infty}\left(\sqrt{\dfrac{4+\frac{1}{x^4}}{\frac{1}{x^3}+2}}+\dfrac{x\sqrt{2-\frac{4}{x^2}}}{x}\right)=\lim\limits_{x\to-\infty}\left(\sqrt{\dfrac{4+\frac{1}{x^4}}{\frac{1}{x^3}+2}}+\sqrt{2-\dfrac{4}{x^2}}\right)$

$=\sqrt{2}+\sqrt{2}=2\sqrt{2}$.