**Câu 14: Tìm các giới hạn sau:**

a). $\lim \dfrac{1+2+3+...+n}{3n^2+1}$  b). $\lim \dfrac{1+3+5+7+...+(2n-1)}{n^2+3n+1}$

c). $\lim \dfrac{2+5+8+...+(3n-1)}{4n^2+1}$  d). $\lim \dfrac{1+2+2^2+2^3+...+2^n}{1+3+3^2+3^3+...+3^n}$

e). $\lim \left[\dfrac{1}{1\sqrt{2}+2\sqrt{1}}+\dfrac{1}{2\sqrt{3}+3\sqrt{2}}+\cdots+\dfrac{1}{n\sqrt{n+1}+(n+1)\sqrt{n}}\right]$

**LỜI GIẢI**

a). $\lim \dfrac{1+2+3+...+n}{3n^2+1}$ $(1)$. Ta có: $1+2+3+...+n=\dfrac{n(n+1)}{2}$ (đã chứng minh bằng phương pháp quy nạp ở chương III).

Vậy $(1) \Leftrightarrow \lim \dfrac{\frac{n(n+1)}{2}}{3n^2+1}=\lim \dfrac{n^2\left(1+\frac{1}{n}\right)}{n^2\left(3+\frac{1}{n^2}\right)}=\lim \dfrac{1+\frac{1}{n^2}}{2\left(3+\frac{1}{n^2}\right)}=\dfrac{1}{6}$.

b). $\lim \dfrac{1+3+5+7+...+(2n-1)}{n^2+3n+1}$ $(1)$

Ta có: $1+3+5+7+...(2n-1)=n^2$ (đã chứng minh bằng phương pháp quy nạp ở chương III).

Vậy $(1) \Leftrightarrow \lim \dfrac{n^2}{n^2+3n+1}=\lim \dfrac{1}{1+\frac{3}{n}+\frac{1}{n^2}}=1$.

c) $\lim \dfrac{2+5+8+...+(3n-1)}{4n^2+1}$ $(1)$

Ta có dãy số: $2;5;8;...;(3n-1)$ là một cấp số cộng với $u_1=2, u_2=5 \Rightarrow d=3$. Số hạng tổng quát:

$u_m=u_1+(m-1)d \ \Leftrightarrow 3n-1=2+(m-1).3$

$\Leftrightarrow 3n-1=3m-1 \Leftrightarrow n=m \Rightarrow$ cấp số cộng có n số hạng.

$S_n=\dfrac{n\left[2u_1+(n-1)d\right]}{2}=\dfrac{n\left[4+(n-1)3\right]}{2}=\dfrac{n(3n+1)}{2}$

$(1) \Leftrightarrow \lim \dfrac{\frac{n(3n+1)}{2}}{4n^2+1}=\lim \dfrac{3+\frac{1}{n}}{2\left(4+\frac{1}{n^2}\right)}=\dfrac{3}{8}$.

d) $\lim \dfrac{1+2+2^2+2^3+...+2^n}{1+3+3^2+3^3+...+3^n}$ $(1)$

Ta có: $1,2,2^2,2^3,...,2^n$ là một dãy số thuộc cấp số nhân, với $u_1=1, q=2$.

Số hạng tổng quát: $u_m=2^n=u_1.q^{m-1} \Leftrightarrow 2^n=2^{m-1} \Leftrightarrow n=m-1 \Rightarrow m=n+1$.

Vậy cấp số nhân này có $(n+1)$ số hạng

$S_m=u_1.\dfrac{1-q^m}{1-q}=\dfrac{1-2^{n+1}}{1-2}=2^{n+1}-1$

Tương tự ta tính được: $1+3+3^2+3^3+...+3^n=\dfrac{3^{n+1}-1}{2}$.

$$(1) \Leftrightarrow \lim \frac{2^{n+1}-1}{\frac{3^{n+1}-1}{2}} = \lim 2\frac{\left(\frac{2}{3}\right)^{n+1} - 3^{\frac{1}{n+1}}}{1-\frac{1}{3^{n+1}}} = 2.0 = 0.$$

e). $L = \lim\left[\frac{1}{1\sqrt{2}+2\sqrt{1}} + \frac{1}{2\sqrt{3}+3\sqrt{2}} + \cdots + \frac{1}{n\sqrt{n+1}+(n+1)\sqrt{n}}\right]$

Ta có: $\frac{1}{n\sqrt{n+1}+(n+1)\sqrt{n}} = \frac{1}{\sqrt{n}\sqrt{n+1}\left(\sqrt{n}+\sqrt{n+1}\right)} = \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n}\sqrt{n+1}} = \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}$

Vậy: $\frac{1}{1\sqrt{2}+2\sqrt{1}} + \frac{1}{2\sqrt{3}+3\sqrt{2}} + \cdots + \frac{1}{n\sqrt{n+1}+(n+1)\sqrt{n}}$

$= \frac{1}{\sqrt{1}} - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} = 1 - \frac{1}{\sqrt{n+1}}$

Vậy $L = \lim\left(1 - \frac{1}{\sqrt{n+1}}\right) = 1$

**Câu 15: Tìm các giới hạn sau:**

a). $\lim \frac{1+\frac{1}{2}+\frac{1}{4}+...+\frac{1}{2^n}}{1+\frac{1}{3}+\frac{1}{9}+...+\frac{1}{3^n}}$ b) $\lim \frac{\sqrt{1+4+7+...+(3n+1)}}{2n^2+\sqrt{n^4+2n+1}}$ c) $\lim\left[\frac{1}{1.3}+\frac{1}{3.5}+...+\frac{1}{(2n-1)(2n+1)}\right]$ d) $\lim\left[\frac{1}{1.3}+\frac{1}{2.4}+...+\frac{1}{n(n+2)}\right]$

**LỜI GIẢI**

a). $\lim \frac{1+\frac{1}{2}+\frac{1}{4}+...+\frac{1}{2^n}}{1+\frac{1}{3}+\frac{1}{9}+...+\frac{1}{3^n}}$ (1)

Ta tính tổng: $1+\frac{1}{2}+\frac{1}{4}+...+\frac{1}{2^n}$. Ta có: $u_1 = 1, u_2 = \frac{1}{2} \Rightarrow q = \frac{u_2}{u_1} = \frac{1}{2}.$

Số hạng tổng quát: $u_m = \frac{1}{2^n} = u_1.q^{m-1}$

$\Leftrightarrow \frac{1}{2^n} = \left(\frac{1}{2}\right)^{m-1} \Leftrightarrow \left(\frac{1}{2}\right)^n = \left(\frac{1}{2}\right)^{m-1} \Leftrightarrow n = m-1 \Leftrightarrow m = n+1.$

$$S_m = u_1.\frac{q^m-1}{q-1} = \frac{\left(\frac{1}{2}\right)^{m+1}-1}{\frac{1}{2}-1} = \left(1-\left(\frac{1}{2}\right)^{n+1}\right).2$$

Tương tự tổng: $1+\frac{1}{3}+\frac{1}{5}+...+\frac{1}{3^n} = \frac{3}{2}\left(1-\left(\frac{1}{3}\right)^{n+1}\right)$

Vậy $(1) \Leftrightarrow \lim \frac{\left(1-\frac{1}{2^{n+1}}\right).2}{\left(1-\frac{1}{3^{n+1}}\right).\frac{3}{2}} = \frac{2}{\frac{3}{2}} = \frac{4}{3}.$

b) $\lim \frac{\sqrt{1+4+7+...+(3n+1)}}{2n^2+\sqrt{n^4+2n+1}}$

Ta tính tổng: $1+4+7+...(3n+1)$. Ta có: $u_1=1, u_2=4 \Rightarrow d=u_2-u_1=3$

Số hạng tổng quát: $u_m=3n+1=u_1+(m-1)d$

$\Leftrightarrow 3n+1=1+(m-1)3 \Leftrightarrow 3n+1=3m-2 \Rightarrow m=n+1$

$S_m=\frac{m}{2}\left[2m+(m-1)d\right]=\frac{n+1}{2}(2+3n)=\frac{(n+1)(3n+2)}{2}$

Vậy: $\lim \frac{\frac{(n+1)(3n+2)}{2}}{2n^2+\sqrt{n^4+2n+1}}=\lim \frac{n^2\left(1+\frac{1}{n}\right)\left(3+\frac{2}{n}\right)}{n^2\left(4+2\sqrt{1+\frac{2}{n^3}+\frac{1}{n^4}}\right)}$

$=\lim \frac{\left(1+\frac{1}{n}\right)\left(3+\frac{2}{n}\right)}{4+2\sqrt{1+\frac{2}{n^3}+\frac{1}{n^4}}}=\frac{3}{6}=\frac{1}{2}.$

c). $\lim \left[\frac{1}{1.3}+\frac{1}{3.5}+...+\frac{1}{(2n-1)(2n+1)}\right]$

$=\lim \frac{1}{2}\left[\frac{2}{1.3}+\frac{2}{3.5}+...+\frac{2}{(2n-1)(2n+1)}\right]$

$=\lim \frac{1}{2}\left[\frac{3-1}{1.3}+\frac{5-3}{3.5}+...+\frac{(2n+1)-(2n-1)}{(2n-1)(2n+1)}\right]$

$=\lim \frac{1}{2}\left(1-\frac{1}{3}+\frac{1}{3}-\frac{1}{5}+...+\frac{1}{2n-1}-\frac{1}{2n+1}\right)$

$=\lim \frac{1}{2}\left(1-\frac{1}{2n+1}\right)=\frac{1}{2}.$

d). $\lim \left[\frac{1}{1.3}+\frac{1}{2.4}+...+\frac{1}{n(n+2)}\right]$

$=\lim \frac{1}{2}\left[\frac{2}{1.3}+\frac{2}{2.4}+...+\frac{2}{n(n+2)}\right]$

$=\lim \frac{1}{2}\left[\frac{3-1}{1.3}+\frac{4-2}{2.4}+...+\frac{n+2-n}{n(n+2)}\right]$

$=\lim \frac{1}{2}\left(1-\frac{1}{3}+\frac{1}{2}-\frac{1}{4}+...+\frac{1}{n}-\frac{1}{n+2}\right)=\lim \frac{1}{2}\left(1-\frac{1}{n+2}\right)=\frac{1}{2}.$

**Câu 16: Tìm các giới hạn sau:**

a). $\lim \left(3n-5-\sqrt{9n^2+1}\right)$ b). $\lim \left(\sqrt{n^2+n+1}-\sqrt[3]{n^3+n^2}\right)$ c). $\lim \left(\sqrt[3]{8n^3+n^2}-\sqrt{4n^2-n+5}\right)$ d). $\lim \frac{(n+1)^4-(n-1)^4}{(n+1)^4+(n-1)^4}$

| | |
|---|---|
| e). $\lim \dfrac{(2n+1)^4-(n-1)^4}{(2n+1)^4+(n-1)^4}$ | f). $\lim \dfrac{\left(\sqrt{n^2+1}+n\right)^2}{\sqrt[3]{n^6+1}}$ |

**LỜI GIẢI**

a). $\lim\left(3n-5-\sqrt{9n^2+1}\right)=\lim\left(3n-\sqrt{9n^2+1}\right)-5$

$=\lim\dfrac{9n^2-(9n^2+1)}{3n+\sqrt{9n^2+1}}-5=\lim\dfrac{-1}{3n+n\sqrt{9+\dfrac{1}{n^2}}}-5$

$=\lim\dfrac{-1}{6n}-5=0-5=-5.$

b). $\lim\left(\sqrt{n^2+n+1}-\sqrt[3]{n^3+n^2}\right)$

$=\lim\left(\sqrt{n^2+n+1}-n+n-\sqrt[3]{n^3+n^2}\right)$

$=\lim\left(\sqrt{n^2+n+1}-n\right)+\lim\left(n-\sqrt[3]{n^3+n^2}\right)$

$=\lim\dfrac{n^2+n+1-n^2}{\sqrt{n^2+n+1}+n}+\lim\dfrac{n^3-(n^3+n^2)}{n^2+n\sqrt[3]{n^3+n^2}+\left(\sqrt[3]{n^3+n^2}\right)^2}$

$=\lim\dfrac{n+1}{\sqrt{n^2\left(1+\dfrac{1}{n}+\dfrac{1}{n^2}\right)}+n}+\lim\dfrac{-n^2}{n^2+n\sqrt[3]{n^3\left(1+\dfrac{1}{n}\right)}+\left(\sqrt[3]{n^3\left(1+\dfrac{1}{n}\right)}\right)^2}$

$=\lim\dfrac{n\left(1+\dfrac{1}{n}\right)}{n^2\left(\sqrt{1+\dfrac{1}{n}+\dfrac{1}{n^2}}+1\right)}+\lim\dfrac{-n^2}{n^2\left[1+\sqrt[3]{1+\dfrac{1}{n}}+\left(\sqrt[3]{1+\dfrac{1}{n}}\right)^2\right]}$

$=\lim\dfrac{1+\dfrac{1}{n}}{\sqrt{1+\dfrac{1}{n}+\dfrac{1}{n^2}}+1}+\lim\dfrac{-1}{1+\sqrt[3]{1+\dfrac{1}{n}}+\left(\sqrt[3]{1+\dfrac{1}{n}}\right)^2}=\dfrac{1}{2}-\dfrac{1}{3}=\dfrac{1}{6}.$

c). $\lim\left(\sqrt[3]{8n^3+n^2}-\sqrt{4n^2-n+5}\right)$

$=\lim\left(\sqrt[3]{8n^3+n^2}-2n+2n-\sqrt{4n^2-n+5}\right)$

$=\lim\left(\sqrt[3]{8n^3+n^2}-2n\right)+\lim\left(2n-\sqrt{4n^2-n+5}\right)$

- Tính $\lim\left(\sqrt[3]{8n^3+n^2}-2n\right)=\lim\dfrac{8n^3+n^2-8n^3}{\left(\sqrt[3]{8n^3+n^2}\right)^2+\sqrt[3]{8n^3+n^2}.2n+4n^2}$

$=\lim\dfrac{n^2}{\left(\sqrt[3]{n^3\left(8+\dfrac{1}{n}\right)}\right)^2+\sqrt[3]{n^3\left(8+\dfrac{1}{n}\right)}.2n+4n^2}$

$$=\lim \frac{n^2}{n^2\left[\left(\sqrt[3]{8+\frac{1}{n}}\right)^2+\sqrt[3]{8+\frac{1}{n}}.2+4\right]}=\lim \frac{1}{\left(\sqrt[3]{8+\frac{1}{n}}\right)^2+\sqrt[3]{8+\frac{1}{n}}.2+4}=\frac{1}{12}.$$

- Tính $\lim\left(2n-\sqrt{4n^2-n+5}\right)$

$$=\lim \frac{4n^2-\left(4n^2-n+5\right)}{2n+\sqrt{4n^2-n+5}}=\lim \frac{n-5}{2n+\sqrt{n^2\left(4-\frac{1}{n}+\frac{5}{n^2}\right)}}$$

$$=\lim \frac{n\left(1-\frac{5}{n}\right)}{n\left(2+\sqrt{4-\frac{1}{n}+\frac{5}{n^2}}\right)}=\lim \frac{1-\frac{5}{n}}{2+\sqrt{4-\frac{1}{n}+\frac{5}{n^2}}}=\frac{1}{4}.$$

Vậy giới hạn cần tìm là: $\frac{1}{12}+\frac{1}{4}=\frac{4}{12}=\frac{1}{3}.$

d). $\lim \frac{(n+1)^4-(n-1)^4}{(n+1)^4+(n-1)^4}=\lim \frac{\left[(n+1)^2-(n-1)^2\right].\left[(n+1)^2+(n-1)^2\right]}{(n+1)^4+(n-1)^4}$

$$=\lim \frac{\left[(n+1)-(n-1)\right]\left[(n+1)+(n-1)\right]\left[(n+1)^2+(n-1)^2\right]}{(n+1)^4+(n-1)^4}$$

$$=\lim \frac{4n\left[n^2\left(1+\frac{1}{n}\right)^2+n^2\left(1-\frac{1}{n}\right)^2\right]}{n^4\left(1+\frac{1}{n}\right)^4+n^4\left(1-\frac{1}{n}\right)^4}=\lim \frac{4n^3\left[\left(1+\frac{1}{n}\right)^2+\left(1-\frac{1}{n}\right)^2\right]}{n^4\left[\left(1+\frac{1}{n^4}\right)+\left(1-\frac{1}{n}\right)^4\right]}=\lim \frac{8}{2n}=0.$$

e). $\lim \frac{(2n+1)^4-(n-1)^4}{(2n+1)^4+(n-1)^4}=\lim \frac{n^4\left(2+\frac{1}{n}\right)-n^4\left(1-\frac{1}{n}\right)^4}{n^4\left(2+\frac{1}{n}\right)^4+n^4\left(1-\frac{1}{n}\right)^4}$

$$=\lim \frac{\left(2+\frac{1}{n}\right)^4-\left(1-\frac{1}{n}\right)^4}{\left(2+\frac{1}{n}\right)^4+\left(1-\frac{1}{n}\right)^4}=\frac{2^4-1^4}{2^4+1^4}=\frac{15}{17}.$$

f). $\lim \frac{\left(\sqrt{n^2+1}+n\right)^2}{\sqrt[3]{n^6+1}}=\lim \frac{\left(n\sqrt{1+\frac{1}{n^2}}+n\right)^2}{\sqrt[3]{n^6\left(1+\frac{1}{n^6}\right)}}=\lim \frac{n^2\left(\sqrt{1+\frac{1}{n^2}}+1\right)^2}{n^2\sqrt[3]{1+\frac{1}{n^6}}}$

$$=\lim\frac{\left(\sqrt{1+\frac{1}{n^2}}+1\right)^2}{\sqrt[3]{1+\frac{1}{n^6}}}=\frac{(1+1)^2}{1}=4.$$

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