$$L=\lim_{x\to1}\frac{x^{100}-2x+1}{x^{50}-2x+1}$$

LỜI GIẢI

$$L=\lim_{x\to1}\frac{x^{100}-x-x+1}{x^{50}-x-x+1}=\lim_{x\to1}\frac{\left(x^{100}-x\right)-\left(x-1\right)}{\left(x^{50}-x\right)-\left(x-1\right)}=\lim_{x\to1}\frac{x\left(x^{99}-1\right)-\left(x-1\right)}{x\left(x^{49}-1\right)-\left(x-1\right)}$$

$$=\lim_{x\to1}\frac{x\left(x-1\right)\left(x^{98}+x^{97}+\cdots+x+1\right)-\left(x-1\right)}{x\left(x-1\right)\left(x^{48}+x^{47}+\cdots+x+1\right)-\left(x-1\right)}=\lim_{x\to1}\frac{\left(x-1\right)\left(x^{99}+x^{98}+\cdots+x^2+x-1\right)}{\left(x-1\right)\left(x^{49}+x^{48}+\cdots+x^2+x-1\right)}$$

$$=\lim_{x\to1}\frac{x^{99}+x^{98}+\cdots+x^2+x-1}{x^{49}+x^{48}+\cdots+x^2+x-1}=\frac{98}{48}=\frac{49}{24}$$

$$L=\lim_{x\to1}\frac{x^{n+1}-\left(n+1\right)x+n}{\left(x-1\right)^2}$$

LỜI GIẢI

Ta có $x^{n+1}-\left(n+1\right)x+n=\left(x^{n+1}-x\right)-\left(nx-n\right)=x\left(x^n-1\right)-n\left(x-1\right)$

$$=x\left(x-1\right)\left(x^{n-1}+x^{n-1}+\cdots+x+1\right)-n\left(x-1\right)=\left(x-1\right)\left(x^n+x^{n-1}+\cdots+x^2+x-n\right)$$

$$=\left(x-1\right)\left(\underbrace{x^n+x^{n-1}+\cdots+x^2+x}_{n\text{ so hang}}-\underbrace{1+1+\cdots+1}_{n\text{ so hang}}\right)$$

$$=\left(x-1\right)\left[\left(x^n-1\right)+\left(x^{n-1}-1\right)+\left(x^2-1\right)+\left(x-1\right)\right]$$

$$=\left(x-1\right)\left[\underbrace{\left(x-1\right)\left(\underbrace{x^{n-1}+x^{n-2}+\cdots+1}_{n}\right)+\left(x-1\right)\left(\underbrace{x^{n-2}+x^{n-3}+\cdots+1}_{n-1}\right)+\cdots+\left(x-1\right)\left(x+1\right)+\left(x-1\right)}_{n}\right]$$

$$=\left(x-1\right)^2\left[\underbrace{\left(\underbrace{x^{n-1}+x^{n-2}+\cdots+1}_{n}\right)+\left(\underbrace{x^{n-2}+x^{n-3}+\cdots+1}_{n-1}\right)+\cdots+\left(x+1\right)+1}_{n}\right]$$

Do đó: $$L=\lim_{x\to1}\frac{\left(x-1\right)^2\left[\underbrace{\left(\underbrace{x^{n-1}+x^{n-2}+\cdots+1}_{n}\right)+\left(\underbrace{x^{n-2}+x^{n-3}+\cdots+1}_{n-1}\right)+\cdots+\left(x+1\right)+1}_{n}\right]}{\left(x-1\right)^2}$$

$$L=\lim_{x\to1}\left[\underbrace{\left(\underbrace{x^{n-1}+x^{n-2}+\cdots+1}_{n}\right)+\left(\underbrace{x^{n-2}+x^{n-3}+\cdots+1}_{n-1}\right)+\cdots+\left(x+1\right)+1}_{n}\right]$$

$$=n+(n-1)+\cdots+2+1=\frac{n\left(n+1\right)}{2}$$

$$\lim_{x\to1}\left(\frac{m}{1-x^m}-\frac{n}{1-x^n}\right),\left(m,n\in\mathbb{N}^*\right)$$

LỜI GIẢI

$$\lim_{x\to1}\left[\left(\frac{m}{1-x^m}-\frac{1}{1-x}\right)-\left(\frac{n}{1-x^n}-\frac{1}{1-x}\right)\right]=\lim_{x\to1}\left(\frac{m}{1-x^m}-\frac{1}{1-x}\right)-\lim_{x\to1}\left(\frac{n}{1-x^n}-\frac{1}{1-x}\right)$$

$$\lim_{x\to1}\left(\frac{m}{1-x^m}-\frac{1}{1-x}\right)=\lim_{x\to1}\frac{m-\left(1+x+x^2+\cdots+x^{m-1}\right)}{1-x^m}$$

$$=\lim_{x\to1}\frac{(1-x)+\left(1-x^2\right)+\cdots+\left(1-x^{m-1}\right)}{1-x^m}$$

$$\lim_{x\to1}\frac{(1-x)\left[1+(1+x)+\cdots+\left(1+x+x^2+\cdots+x^{m-2}\right)\right]}{(1-x)\left(1+x+x^2+\cdots+x^{m-1}\right)}$$

$$\lim_{x\to1}\frac{1+(1+x)+\cdots+\left(1+x+x^2+\cdots+x^{m-2}\right)}{1+x+x^2+\cdots+x^{m-1}}=\frac{1+2+\cdots+m-1}{m}=\frac{m-1}{2}$$

Tương tự $\lim_{x\to1}\left(\frac{n}{1-x^n}-\frac{1}{1-x}\right)=\frac{n-1}{2}$

Vậy $\lim_{x\to1}\left(\frac{m}{1-x^m}-\frac{n}{1-x^n}\right)=\frac{m-1}{2}-\frac{n-1}{2}=\frac{m-n}{2}$

Ví dụ: Tìm các giới hạn sau:

a). $\lim_{x\to1}\frac{\sqrt{3x+1}\sqrt[3]{2-x}-2}{x-1}$ b). $\lim_{x\to0}\frac{\sqrt{4+x}\sqrt[3]{8+3x}-4}{x^2+x}$

c). $\lim_{x\to0}\frac{\sqrt{1.2x+1}\sqrt[3]{2.3x+1}\sqrt[4]{3.4x+1}-1}{x}$

LỜI GIẢI

a). $\lim_{x\to1}\frac{\sqrt{3x+1}\left(\sqrt[3]{2-x}-1\right)+\sqrt{3x+1}-2}{x-1}=\lim_{x\to1}\frac{\sqrt{3x+1}\left(\sqrt[3]{2-x}-1\right)}{x-1}+\lim_{x\to1}\frac{\sqrt{3x+1}-2}{x-1}$

- Tính $\lim_{x\to1}\frac{\sqrt{3x+1}\left(\sqrt[3]{2-x}-1\right)}{x-1}=\lim_{x\to1}\frac{\sqrt{3x+1}(2-x-1)}{(x-1)\left[\left(\sqrt[3]{2-x}\right)^2+\sqrt[3]{2-x}+1\right]}$

$$=\lim_{x\to1}\frac{-\sqrt{3x+1}}{\left(\sqrt[3]{2-x}\right)^2+\sqrt[3]{2-x}+1}=-\frac{2}{3}$$

- Tính $\lim_{x\to1}\frac{\sqrt{3x+1}-2}{x-1}=\lim_{x\to1}\frac{3x+1-4}{(x-1)\left(\sqrt{3x+1}+2\right)}=\lim_{x\to1}\frac{3}{\sqrt{3x+1}+2}=\frac{3}{4}$.

Vậy $\lim_{x\to1}\frac{\sqrt{3x+1}\sqrt[3]{2-x}-2}{x-1}=-\frac{2}{3}+\frac{3}{4}=\frac{1}{12}$.

b). $\lim_{x\to0}\frac{\sqrt[3]{8+3x}\left(\sqrt{4+x}-2\right)+2\sqrt[3]{8+3x}-4}{x^2+x}=\lim_{x\to0}\frac{\sqrt[3]{8+3x}\left(\sqrt{4+x}-2\right)}{x^2+x}+\lim_{x\to0}\frac{2\sqrt[3]{8+3x}-4}{x^2+x}$

- Tính $\lim_{x\to0}\frac{\sqrt[3]{8+3x}\left(\sqrt{4+x}-2\right)}{x^2+x}=\lim_{x\to0}\frac{\sqrt[3]{8+3x}.x}{x(x+1)\left(\sqrt{4+x}+2\right)}=\lim_{x\to0}\frac{\sqrt[3]{8+3x}}{(x+1)\left(\sqrt{4+x}+2\right)}=\frac{1}{2}$

- Tính $\lim\limits_{x\to 0}\dfrac{2\sqrt[3]{8+3x}-4}{x^2+x}=2\lim\limits_{x\to 0}\dfrac{8+3x-8}{x(x+1)\left[\left(\sqrt[3]{8+3x}\right)^2+2\sqrt[3]{8+3x}+4\right]}$

$=2\lim\limits_{x\to 0}\dfrac{3}{(x+1)\left[\left(\sqrt[3]{8+3x}\right)^2+2\sqrt[3]{8+3x}+4\right]}=\dfrac{1}{2}$

Vậy $\lim\limits_{x\to 0}\dfrac{\sqrt{4+x}\sqrt[3]{8+3x}-4}{x^2+x}=\dfrac{1}{2}+\dfrac{1}{2}=1$

c). $L=\lim\limits_{x\to 0}\dfrac{\sqrt{1.2x+1}\sqrt[3]{2.3x+1}\sqrt[4]{3.4x+1}-1}{x}=\lim\limits_{x\to 0}\dfrac{\left(\sqrt{2x+1}-1\right)\sqrt[3]{2.3x+1}\sqrt[4]{3.4x+1}}{x}$

$+\lim\limits_{x\to 0}\dfrac{\left(\sqrt[3]{2.3x+1}-1\right)\sqrt[4]{3.4x+1}}{x}+\lim\limits_{x\to 0}\dfrac{\sqrt[4]{3.4x+1}-1}{x}$

Ta chứng minh được $\lim\limits_{x\to 0}\dfrac{\sqrt[n]{ax+1}-1}{x}=\dfrac{a}{n}\left(a\neq 0, n\in \mathbb{N}^*\right)$