DẠNG 2: u_n **là một phân thức hữu tỉ dạng** $u_n=\frac{P(n)}{Q(n)}$ **(trong đó** $P(n),Q(n)$ **là các biểu thức chứa căn của n).**

Ví dụ: Tìm giới hạn của dãy (u_n) **biết:**

a). $u_n=\frac{\sqrt{4n^2-n+1}-n}{\sqrt{9n^2+3n}}$ b). $u_n=\frac{\sqrt{2n+1}-\sqrt{n+3}}{\sqrt{4n-5}}$

c). $u_n=\frac{\sqrt{4n^2-1}+\sqrt[3]{8n^3+2n^2-3}}{\sqrt{16n^2+4n}-\sqrt[4]{n^4+1}}$ d). $u_n=\frac{\sqrt{n^3+n}+\sqrt[3]{n^3+3n}}{\sqrt[4]{16n^4+1}}$ e).

LỜI GIẢI

a). $u_n=\frac{\sqrt{4n^2-n+1}-n}{\sqrt{9n^2+3n}}=\frac{\sqrt{n^2\left(\frac{4n^2-n+1}{n^2}\right)}-n}{\sqrt{n^2\left(\frac{9n^2+3n}{n^2}\right)}}=\frac{n\sqrt{4-\frac{1}{n}+\frac{1}{n^2}}-n}{n\sqrt{9+\frac{3}{n}}}=\frac{\sqrt{4-\frac{1}{n}+\frac{1}{n^2}}-1}{\sqrt{9+\frac{3}{n}}}$. Vì có $\lim\frac{1}{n}=0$, $\lim\frac{1}{n^2}=0$, và $\lim\frac{3}{n}=0$. Nên $\lim u_n=\frac{\sqrt{4-0+0}-1}{\sqrt{9+0}}=\frac{1}{3}$.

b). $u_n=\frac{\sqrt{2n+1}-\sqrt{n+3}}{\sqrt{4n-5}}=\frac{\sqrt{n\left(\frac{2n+1}{n}\right)}-\sqrt{n\left(\frac{n+3}{n}\right)}}{\sqrt{n\left(\frac{4n-5}{n}\right)}}=\frac{\sqrt{n}.\sqrt{2+\frac{1}{n}}-\sqrt{n}.\sqrt{1+\frac{3}{n}}}{\sqrt{n}.\sqrt{4-\frac{5}{n}}}=\frac{\sqrt{2+\frac{1}{n}}-\sqrt{1+\frac{3}{n}}}{\sqrt{4-\frac{5}{n}}}$. Vì có $\lim\frac{1}{n}=0$, $\lim\frac{3}{n}=0$ và $\lim\frac{5}{n}=0$.

Từ đó có $\lim u_n=\frac{\sqrt{2+0}-\sqrt{1+0}}{\sqrt{4-0}}=\frac{\sqrt{2}-1}{2}$.

c). Ta có $u_n=\frac{\sqrt{4n^2-1}+\sqrt[3]{8n^3+2n^2-3}}{\sqrt{16n^2+4n}-\sqrt[4]{n^4+1}}=\frac{\sqrt{n^2\left(\frac{4n^2-1}{n^2}\right)}+\sqrt[3]{n^3\left(\frac{8n^3+2n^2-3}{n^3}\right)}}{\sqrt{n^2\left(\frac{16n^2+4n}{n^2}\right)}+\sqrt[4]{n^4\left(\frac{n^4+1}{n^4}\right)}}$

$=\frac{n.\sqrt{4-\frac{1}{n^2}}+n.\sqrt[3]{8+\frac{2}{n}-\frac{3}{n^3}}}{n.\sqrt{16+\frac{4}{n}}+n.\sqrt{1+\frac{1}{n^4}}}=\frac{\sqrt{4-\frac{1}{n^2}}+\sqrt[3]{8+\frac{2}{n}-\frac{3}{n^3}}}{\sqrt{16+\frac{4}{n}}+\sqrt{1+\frac{1}{n^4}}}$. Vì có $\lim\frac{1}{n^2}=0$, $\lim\frac{2}{n}=0$, $\lim\frac{3}{n^3}=0$, $\lim\frac{4}{n}=0$ và $\lim\frac{1}{n^4}=0$. Từ đó suy ra $\lim u_n=\frac{\sqrt{4-0}+\sqrt[3]{8+0-0}}{\sqrt{16+0}+\sqrt{1+0}}=\frac{4}{5}$.

d). Ta có $u_n=\frac{\sqrt{n^2+n}+\sqrt[3]{n^3+3n}}{\sqrt[4]{16n^4+1}}=\frac{\sqrt{n^2\left(\frac{n^2+n}{n^2}\right)}+\sqrt[3]{n^3\left(\frac{n^3+3n}{n^3}\right)}}{\sqrt[4]{n^4\left(\frac{16n^4+1}{n^4}\right)}}$

$$=\frac{n.\sqrt{1+\frac{1}{n}}+n.\sqrt[3]{1+\frac{3}{n^2}}}{n.\sqrt[4]{16+\frac{1}{n^4}}}=\frac{\sqrt{1+\frac{1}{n}}+\sqrt[3]{1+\frac{3}{n^2}}}{\sqrt[4]{16+\frac{1}{n^4}}}$$. Vì có $\lim\frac{1}{n}=0,\ \lim\frac{3}{n^2}=0,$ và $\lim\frac{1}{n^4}=0$. Nên

$$\lim u_n=\frac{\sqrt{1+0}+\sqrt[3]{1+0}}{\sqrt[4]{16+0}}=\frac{1}{2}.$$

DẠNG 3: u_n **là một phân thức hữu tỉ dạng** $u_n=\frac{P(n)}{Q(n)}$ **(trong đó** $P(n),Q(n)$ **là các biểu thức chứa hàm mũ** a^n,b^n,c^n **,…. Chia cả tử và mẫu cho** a^n **với a là cơ số lớn nhất).**

Ví dụ: Tìm giới hạn của dãy (u_n) **biết:**

a). $u_n=\frac{2^n+4^n}{4^n-3^n}$ b). $u_n=\frac{3.2^n-5^n}{5.4^n+6.5^n}$ c). $u_n=\frac{4^{n+2}+6^{n+1}}{5^{n-1}+2.6^{n+3}}$

d). $u_n=\frac{\sqrt{2}^{n+2}+1}{3^{\frac{n}{2}}+2}$ e). $u_n=\frac{(-3)^n-4.5^{n+1}}{2.4^n+3.5^n}$ f). $u_n=\frac{2^n-3^n+4.5^{n+2}}{2^{n+1}+3^{n+2}+5^{n+1}}$

LỜI GIẢI

a).Ta có $u_n=\frac{2^n+4^n}{4^n-3^n}=\frac{\frac{2^n+4^n}{4^n}}{\frac{4^n-3^n}{4^n}}=\frac{\frac{2^n}{4^n}+\frac{4^n}{4^n}}{\frac{4^n}{4^n}-\frac{3^n}{4^n}}=\frac{\left(\frac{2}{4}\right)^n+1}{1-\left(\frac{3}{4}\right)^n}$. Ta có $\lim\left(\frac{2}{4}\right)^n=0$ và $\lim\left(\frac{3}{4}\right)^n=0$. Nên

$$\lim u_n=\frac{0+1}{1-0}=1.$$

b). Ta có $u_n=\frac{3.2^n-5^n}{5.4^n+6.5^n}=\frac{\frac{3.2^n-5^n}{5^n}}{\frac{5.4^n+6.5^n}{5^n}}=\frac{\frac{3.2^n}{5^n}-\frac{5^n}{5^n}}{\frac{5.4^n}{5^n}+\frac{6.5^n}{5^n}}=\frac{3\left(\frac{2}{5}\right)^n-1}{5\left(\frac{4}{5}\right)^n+6}$. Ta có $\lim\left(\frac{2}{5}\right)^n=0$ và $\lim\left(\frac{4}{5}\right)^n=0$.

Do đó $\lim u_n=\frac{3.0-1}{5.0+6}=-\frac{1}{6}$.

c). Ta có $u_n=\frac{4^{n+2}+6^{n+1}}{5^{n-1}+2.6^{n+3}}=\frac{4^n.4^2+6^n.6}{5^n.5^{-1}+2.6^n.6^3}=\frac{\frac{4^n.4^2+6^n.6}{6^n}}{\frac{5^n.5^{-1}+2.6^n.6^3}{6^n}}=\frac{\frac{4^n.4^2}{6^n}+\frac{6^n.6}{6^n}}{\frac{5^n.5^{-1}}{6^n}+\frac{2.6^n.6^3}{6^n}}$

$=\frac{4^2\left(\frac{4}{6}\right)^n+6}{5^{-1}\left(\frac{5}{6}\right)^n+2.6^3}$. Ta có $\lim\left(\frac{4}{6}\right)^n=0$ và $\lim\left(\frac{5}{6}\right)^n=0$.

Do đó $\lim u_n=\frac{4^2.0+6}{5^{-1}.0+2.6^3}=\frac{1}{72}$.

d). Ta có $u_n = \dfrac{\sqrt{2}^{n+2}+1}{3^{\frac{n}{2}}+2} = \dfrac{2^{\frac{n}{2}+1}+1}{3^{\frac{n}{2}}+2} = \dfrac{2.2^{\frac{n}{2}}+1}{3^{\frac{n}{2}}+2} = \dfrac{\dfrac{2.2^{\frac{n}{2}}+1}{3^{\frac{n}{2}}}}{\dfrac{3^{\frac{n}{2}}+2}{3^{\frac{n}{2}}}} = \dfrac{2.\left(\dfrac{2}{3}\right)^{\frac{n}{2}}+\dfrac{1}{3^{\frac{n}{2}}}}{1+\dfrac{2}{3^{\frac{n}{2}}}}$. Vì $\left|\dfrac{2}{3}\right|<1 \Rightarrow \lim\left(\dfrac{2}{3}\right)^{\frac{n}{2}}=0$,

$\lim\dfrac{1}{3^{\frac{n}{2}}}=0$ và $\lim\dfrac{2}{3^{\frac{n}{2}}}=0$. Do đó $\lim u_n=\dfrac{2.0+0}{1+0}=0$.

e). Ta có : $u_n=\dfrac{(-3)^n-4.5^{n+1}}{2.4^n+3.5^n}=\dfrac{(-3)^n-20.5^n}{2.4^n+3.5^n}=\dfrac{\dfrac{(-3)^n-20.5^n}{5^n}}{\dfrac{2.4^n+3.5^n}{5^n}}=\dfrac{\dfrac{(-3)^n}{5^n}-20.\dfrac{5^n}{5^n}}{2.\dfrac{4^n}{5^n}+3.\dfrac{5^n}{5^n}}=\dfrac{\left(-\dfrac{3}{5}\right)^n-20}{2.\left(\dfrac{4}{5}\right)^n+3}$, mà

$\lim\left(-\dfrac{3}{5}\right)^n=0$ và $\lim\left(\dfrac{4}{5}\right)^n=0$. Do đó $\lim u_n=\dfrac{0-20}{2.0+3}=-\dfrac{20}{3}$.

f). Ta có $u_n=\dfrac{2^n-3^n+4.5^{n+2}}{2^{n+1}+3^{n+2}+5^{n+1}}=\dfrac{2^n-3^n+100.5^n}{2.2^n+9.3^n+5.5^n}=\dfrac{\dfrac{2^n-3^n+100.5^n}{5^n}}{\dfrac{2.2^n+9.3^n+5.5^n}{5^n}}$

$=\dfrac{\dfrac{2^n}{5^n}-\dfrac{3^n}{5^n}+100.\dfrac{5^n}{5^n}}{2.\dfrac{2^n}{5^n}+9.\dfrac{3^n}{5^n}+5.\dfrac{5^n}{5^n}}=\dfrac{\left(\dfrac{2}{5}\right)^n-\left(\dfrac{3}{5}\right)^n+100}{2.\left(\dfrac{2}{5}\right)^n+9.\left(\dfrac{3}{5}\right)^n+5}$. Vì $\lim\left(\dfrac{2}{5}\right)^n=0$ và $\lim\left(\dfrac{3}{5}\right)^n=0$ nên

$\lim u_n=\dfrac{0-0+100}{2.0+9.0+5}=20$.

DẠNG 4 : Nhân lượng liên hợp:

PHƯƠNG PHÁP : Sử dụng các công thức nhân lượng liên hợp sau:

- $a^2-b^2=(a+b)(a-b)\rightarrow\left[\begin{array}{l} a-b=\dfrac{a^2-b^2}{a+b} \\ a+b=\dfrac{a^2-b^2}{a-b}\end{array}\right.$
- $a-b=\dfrac{a^3-b^3}{a^2+ab+b^2}$ • $a+b=\dfrac{a^3+b^3}{a^2-ab+b^2}$.
- $\sqrt[3]{a}-b=\dfrac{\left(\sqrt[3]{a}-b\right)\left[\left(\sqrt[3]{a}\right)^2+\sqrt[3]{a}.b+b^2\right]}{\left(\sqrt[3]{a}\right)^2+\sqrt[3]{a}.b+b^2}=\dfrac{a-b^3}{\left(\sqrt[3]{a}\right)^2+\sqrt[3]{a}.b+b^2}$.
- $\sqrt[3]{a}+b=\dfrac{\left(\sqrt[3]{a}+b\right)\left[\left(\sqrt[3]{a}\right)^2-\sqrt[3]{a}.b+b^2\right]}{\left(\sqrt[3]{a}\right)^2-\sqrt[3]{a}.b+b^2}=\dfrac{a+b^3}{\left(\sqrt[3]{a}\right)^2-\sqrt[3]{a}.b+b^2}$

- $a-\sqrt[3]{b}=\dfrac{\left(a-\sqrt[3]{b}\right)\left[a^2+a.\sqrt[3]{b}+\left(\sqrt[3]{b}\right)^2\right]}{a^2+a.\sqrt[3]{b}+\left(\sqrt[3]{b}\right)^2}=\dfrac{a^3-b}{a^2+a.\sqrt[3]{b}+\left(\sqrt[3]{b}\right)^2}$
- $a+\sqrt[3]{b}=\dfrac{\left(a+\sqrt[3]{b}\right)\left[a^2-a.\sqrt[3]{b}+\left(\sqrt[3]{b}\right)^2\right]}{a^2-a.\sqrt[3]{b}+\left(\sqrt[3]{b}\right)^2}=\dfrac{a^3+b}{a^2-a.\sqrt[3]{b}+\left(\sqrt[3]{b}\right)^2}$
- $\sqrt[3]{a}-\sqrt[3]{b}=\dfrac{\left(\sqrt[3]{a}-\sqrt[3]{b}\right)\left[\left(\sqrt[3]{a}\right)^2+\sqrt[3]{a}.\sqrt[3]{b}+\left(\sqrt[3]{b}\right)^2\right]}{\left(\sqrt[3]{a}\right)^2+\sqrt[3]{a}.\sqrt[3]{b}+\left(\sqrt[3]{b}\right)^2}=\dfrac{a-b}{\left(\sqrt[3]{a}\right)^2+\sqrt[3]{a}.\sqrt[3]{b}+\left(\sqrt[3]{b}\right)^2}$.
- $\sqrt[3]{a}+\sqrt[3]{b}=\dfrac{\left(\sqrt[3]{a}+\sqrt[3]{b}\right)\left[\left(\sqrt[3]{a}\right)^2-\sqrt[3]{a}.\sqrt[3]{b}+\left(\sqrt[3]{b}\right)^2\right]}{\left(\sqrt[3]{a}\right)^2-\sqrt[3]{a}.\sqrt[3]{b}+\left(\sqrt[3]{b}\right)^2}=\dfrac{a+b}{\left(\sqrt[3]{a}\right)^2-\sqrt[3]{a}.\sqrt[3]{b}+\left(\sqrt[3]{b}\right)^2}$