DẠNG 4: Các giới hạn đặc biệt

Nhắc lại:

- $\underbrace{1+1+1+\cdots+1}_{n}=n$
- $a^3-b^3=(a-b)\left(\underbrace{a^2+ab+b^2}_{\text{3 so hang}}\right)$
- $a^n-b^n=(a-b)\left(\underbrace{a^{n-1}+a^{n-2}b+a^{n-3}b^2+\cdots+ab^{n-2}+b^{n-1}}_{\text{n so hang}}\right)$

Tìm $L=\lim\limits_{x\to 0}\dfrac{\sqrt[n]{1+ax}-1}{x}$

LỜI GIẢI

Cách giải: Đặt $t=\sqrt[n]{1+ax}\Rightarrow t^n=1+ax\Rightarrow x=\dfrac{t^n-1}{a}$

Ta có khi $x\to 0$ thì $t\to 1$.

Khi đó $L=\lim\limits_{x\to 1}\dfrac{a(t-1)}{t^n-1}$

$$=\lim_{x\to 1}\frac{a(t-1)}{(t-1)(t^{n-1}+t^{n-2}+\cdots+t+1)}=\lim_{x\to 1}\frac{a}{t^{n-1}+t^{n-2}+\cdots+t+1}=\frac{a}{n}$$

Vậy $L=\lim\limits_{x\to 0}\dfrac{\sqrt[n]{1+ax}-1}{x}=\dfrac{a}{n}$

$L=\lim\limits_{x\to 0}\dfrac{\sqrt[n]{1+ax}-\sqrt[m]{1+bx}}{x}$

LỜI GIẢI

$L=\lim\limits_{x\to 0}\dfrac{\sqrt[n]{1+ax}-1+1-\sqrt[m]{1+bx}}{x}=\lim\limits_{x\to 0}\dfrac{\sqrt[n]{1+ax}-1}{x}-\lim\limits_{x\to 0}\dfrac{\sqrt[m]{1+bx}-1}{x}=\dfrac{a}{n}-\dfrac{b}{m}$ (áp dụng kết quả bài kế trên).

$L=\lim\limits_{x\to 0}\dfrac{\sqrt[n]{1+ax}-1}{\sqrt[m]{1+bx}-1}\ (ab\neq 0)$

LỜI GIẢI

$L=\lim\limits_{x\to 0}\dfrac{\sqrt[n]{1+ax}-1}{x}\cdot\dfrac{x}{\sqrt[m]{1+bx}-1}=\dfrac{a}{n}\cdot\dfrac{m}{b}=\dfrac{am}{bn}$ (áp dụng kết quả bài trên).

$L=\lim\limits_{x\to 0}\dfrac{\sqrt[n]{1+ax}-\sqrt[m]{1+bx}}{\sqrt{1+x}-1}$

LỜI GIẢI

$$L=\lim_{x\to 0}\frac{\sqrt[n]{1+ax}-1+1-\sqrt[m]{1+bx}}{x}\cdot\frac{x}{\sqrt{1+x}-1}=\lim_{x\to 0}\left(\frac{\sqrt[n]{1+ax}-1}{x}-\frac{\sqrt[m]{1+bx}-1}{x}\right)\left(\sqrt{1+x}+1\right)$$

$$=2\left(\frac{a}{n}-\frac{b}{m}\right)$$

$L=\lim\limits_{x\to 1}\dfrac{\sqrt[m]{x}-1}{\sqrt[n]{x}-1}$

LỜI GIẢI

Đặt $t=\sqrt[mn]{x}\Rightarrow x=t^{mn}$, vậy $\sqrt[m]{x}=t^{n},\sqrt[n]{x}=t^{m}$

$$L=\lim_{t\to 1}\frac{t^{n}-1}{t^{m}-1}=\lim_{t\to 1}\frac{(t-1)\left(t^{n-1}+t^{n-2}+\cdots+t+1\right)}{(t-1)\left(t^{m-1}+t^{m-2}+\cdots+t+1\right)}\lim_{t\to 1}\frac{t^{n-1}+t^{n-2}+\cdots+t+1}{t^{m-1}+t^{m-2}+\cdots+t+1}=\frac{n}{m}.$$

$$\boxed{L=\lim_{x\to 1}\frac{x+x^{2}+x^{3}+\cdots+x^{n}-n}{x+x^{2}+x^{3}+\cdots+x^{m}-m}}$$

LỜI GIẢI

Ta có: $x+x^{2}+x^{3}+\cdots+x^{n}-n=(x-1)+\left(x^{2}-1\right)+\left(x^{3}-1\right)+\cdots+\left(x^{n}-1\right)$

$=(x-1)+(x-1)(x+1)+(x-1)\left(x^{2}+x+1\right)+\cdots+(x-1)\left(x^{n-1}+x^{n-1}+\cdots+1\right)$

$=(x-1)\left[1+(x+1)+\left(x^{2}+x+1\right)+\cdots+\left(x^{n-1}+x^{n-1}+\cdots+1\right)\right]$

Tương tự: : $x+x^{2}+x^{3}+\cdots+x^{m}-m=(x-1)+\left(x^{2}-1\right)+\left(x^{3}-1\right)+\cdots+\left(x^{m}-1\right)$

$=(x-1)+(x-1)(x+1)+(x-1)\left(x^{2}+x+1\right)+\cdots+(x-1)\left(x^{m-1}+x^{m-1}+\cdots+1\right)$

$=(x-1)\left[1+(x+1)+\left(x^{2}+x+1\right)+\cdots+\left(x^{m-1}+x^{m-1}+\cdots+1\right)\right]$

Vậy $L=\lim_{x\to 1}\dfrac{(x-1)\left[1+(x+1)+\left(x^{2}+x+1\right)+\cdots+\left(x^{n-1}+x^{n-1}+\cdots+1\right)\right]}{(x-1)\left[1+(x+1)+\left(x^{2}+x+1\right)+\cdots+\left(x^{m-1}+x^{m-1}+\cdots+1\right)\right]}$

$$=\lim_{x\to 1}\frac{1+(x+1)+\left(x^{2}+x+1\right)+\cdots+\left(x^{n-1}+x^{n-1}+\cdots+1\right)}{1+(x+1)+\left(x^{2}+x+1\right)+\cdots+\left(x^{m-1}+x^{m-1}+\cdots+1\right)}$$

$$=\frac{1+2+3+\cdots+n}{1+2+3+\cdots+m}=\frac{\frac{n(n+1)}{2}}{\frac{m(m+1)}{2}}=\frac{n(n+1)}{m(m+1)}.$$